Fundamental solutions and matrix exponentials

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Review We consider a homogeneous system of linear differential equations of the form

$$\mathbf{x}' = A\mathbf{x}, \tag{1}$$

where A is an $n \times n$ matrix of continuous functions and $\mathbf{x}$ is a column vector of differentiable functions.

- We know that the set of solutions is a vector space of dimension n. A basis for this space is sometimes called a *fundamental solution set* to the equation.
- If $(\mathbf{x}_1, \cdots \mathbf{x}_n)$ is such a basis, then the matrix $\mathbf{X}$ whose columns are the vectors $\mathbf{x}_i$ is sometimes called a *fundamental matrix* for the equation.

Applications This construction is useful for the following reasons. **Theorem:** Let $\mathbf{X}$ be a fundamental matrix for the equation (1) above.

- If $\mathbf{v}$ is any vector in $\mathbf{R}^n$, then the vector

  $$\mathbf{y} := \mathbf{X}\mathbf{v}$$

  is a solution to (1), and every solution is of this form.
- If C is an invertible matrix, then

  $$\tilde{\mathbf{X}} := \mathbf{X}C$$

  is also a fundamental matrix for (1).
- In particular,

  $$\tilde{\mathbf{X}} := \mathbf{X}\mathbf{X}(0)^{-1}$$

  is the fundamental matrix with $\tilde{\mathbf{X}}(0) = I_n$.
- If $\mathbf{x}_0 \in \mathbf{R}^n$, then

  $$\mathbf{x} := \tilde{\mathbf{X}}\mathbf{x}_0$$

  is the solution to equation (1) satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

Matrix Exponentials

When A is *constant*, matrix exponentials can be used to find a fundamental matrix for the equation (1).

Theorem: If A is any $n \times n$ matrix (real or complex)

$$\exp A := e^A := I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots$$

- The series above converges, for every matrix A.
- If A and B commute, then $\exp(A + B) = \exp(A) \exp B$.
- If S is invertible, then $S(\exp A)S^{=1} = \exp(SAS^{-1})$.
- If A is constant, $\mathbf{X}(t) := \exp(tA)$, then $\mathbf{X}'(t) = A \exp(tA) = A\mathbf{X}(t)$.

In particular, if A is constant, then $\mathbf{X}(t) := \exp(tA)$ is a fundamental matrix for the equation (1).

Example: Diagonalizable matrices
If $D := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is a diagonal matrix, then e^{tD} is easy to compute:

$$e^{tD} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}$$

Example: Complex eigenvalues This formula works even if the eigenvalues λ_i are complex, if we apply Euler's formula. Sometimes it is easier to work directly. For example, suppose

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then $A^2 = -I$ and hence

$$\begin{aligned} e^{tA} &= I + \frac{tA}{1!} + \frac{t^2 A}{2!} + \frac{t^3 A}{3!} + \cdots \\ &= \left(I - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \right) + A \left(\frac{t}{1!} - \frac{t^3}{3!} + \cdots \right) \\ &= \cos t I + \sin t A \end{aligned}$$

Thus

$$e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Example: nondiagonalizalbe matrices

Consider the matrix $N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $N^2 = 0$, and hence

$$e^{tN} := I + tN + \frac{t^2N^2}{2!} + \cdots = I + tN.$$

Thus

$$e^{tN} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Here is a more subtle example. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$, so $f_A(t) = t^2+2t+1 = (t+1)^2$. Then -1 is the only eigenvalue, Then $Eig_{-1}(A) = NS\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, which is one dimensional, with basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now we can do the following. Let $N := \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, and observe that $N^2 = 0$ and that $A = D + N$, where $D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $DN = ND$. Hence

$$\begin{aligned} e^{tA} &= e^{tD+tN} \\ &= e^{tD}e^{tN} \\ &= e^{-t}(I+tN) \end{aligned}$$

For 2×2 matrices, this always works. Here is a brief explanation.

Theorem: Let A be a 2×2 matrix with only a single eigenvalue λ. Let $D := \lambda I$ and $N := A - D$. Then $A = D + N$, $DN = ND$, and $N^2 = 0$. It follows that $e^{tA} = e^{t\lambda}(I + tN)$.

Proof: The only thing that requires proof here is the fact that $N^2 = 0$. Notice first that since λ is the only eigenvalue of A, $f_A(t) = (t-\lambda)^2$. Now we use the Cayley-Hamilton theorem, which says that if we plug A into its characteristic polynomial and compute using matrix algebra, the answer is 0. In this case, we get $(A - \lambda I)^2 = 0$, $N^2 = 0$, which is what we want.

The Cayley-Hamilton theorem is strange and important; it is worth looking more closely.

Theorem (Cayley-Hamilton): Let A be an $n \times n$ matrix, and let $f_A(t)$ be its characteristic polynomial. Then $f_A^(A)$ is the zero matrix.

Proof (when $n = 2$). This theorem is difficult in general, but we can do the case $n = 2$ by hand. Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $f_A(t) = t^2 - (a+d)t + ad - bc$. Hence:

$$\begin{aligned} f_A(A) &= A^2 - (a+d)A + (ad-bc)I \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)I \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+dc & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= 0 \end{aligned}$$

A miracle! If you want to see why it is true for larger n, and how to use it to compute e^A in general, take Math 110.