Images, Kernels, and Linear Subspaces

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The image of a function

- Let $f: V \to W$ be a function.

- Given w in W, when can we solve the equation

$$f(v) = w\ ?$$

- **Definition**: The image of f is the set of all w in W such that there is at least one v such that $f(v) = w$.

- Thus $\mathrm{Im}(f)$ is a subset of W: $\mathrm{Im}(f) \subseteq W$.

- **Theorem**: Let T: $R^m \to R^n$ be a linear transformation.
  - The zero vector of R^n belongs to Im(T).
  - If Y and Y' belong to Im(T), then so does Y+Y'.
  - If Y belongs to Im(T), then so does cY, for any number c.

- **Corollary**: The image of T is not empty and is closed under the formation of linear combinations.

Proof:

- T takes the zero vector of R^m to the zero vector of R^n. Hence the zero vector of R^n is in the image of T.
- If Y and Y′ belong to the image,
  - then there exist X and X′ with T(X) = Y and T(X′) = Y′,
  - then T(X+X′) = T(X) + T(X′) = Y+ Y′.
  - So Y+Y′ is in the image too.
- Similarly, if T(X) = Y, then T(cX) = cY.

Spans and the column space

- **Theorem**: If $A \in M_{nm}$, then $\mathrm{Im}(T_A)$ is the set of vectors which can be written as a linear combination of the columns of A.
- **Definition**: The span of a sequence $(v_1, v_2, ..., v_m)$ of vectors is the set of all vectors which can be written as some linear combination of $(v_1, v_2, ..., v_m)$
- Thus the image of T_A is the span of the columns of A.

Proof of the theorem:

- For each j, $T_A(e_j)$ is the jth column of A.
  - Thus every column of A belongs to the image of T_A.
  - Hence every linear combination of the columns of A belongs to the image.
- If Y belongs to the image, then $Y = T_A(X)$ for some X. But then
  - $Y = x_1 C_1(A) + x_2C_2(A) + \ldots x_m C_m(A)$,
  - so Y belongs to the column space of A.

- **Theorem**: Let T: $R^m \rightarrow R^n$ be a linear transformation. Then the following are equivalent:
  - T is surjective
  - $Im(T) = R^n$.
  - rank(A) = n, where $T = T_A$.

The kernel of a linear transformation

- **Definition**: If $T: R^m \rightarrow R^n$ is a linear transformation, the kernel of T is the set of all X such that $T(X) = 0$.

- **Theorem**: Let T: $R^m \to R^n$ be a linear transformation.
  - The zero vector of R^m belongs to Ker(T).
  - If X and X' belong to Ker(T), then so does X+X'.
  - If X belongs to Ker(T), then so does cX, for any number c.

- **Corollary**: The kernel of a linear transformation is not empty and is closed under formation of linear combinations.

- Proof of theorem:

- if X and X' are in the kernel,

  - $T(X+X') = T(X) + T(X') = 0 + 0 = 0.$

  - $T(cX) = cT(X) = c\,0 = 0.$

- **Theorem**: Let $T: R^m \to R^n$ be a linear transformation. Then the following are equivalent:
  - T is injective
  - $Ker(T) = \{0\}$
  - $rank(A) = m$, where $T = T_A$.

- **Definition**: A subset W of R^m is a linear subspace if it satisfies the following conditions:
  - it contains the zero vector,
  - if X and X′ belong to W, so does X+X′,
  - if X belongs to W and c is any number, cX belongs to W.

- **Equivalently**: A subset W of R^m is a linear subspace if is not empty and is closed under formation of linear combinations.

- Thus: If T is a linear transformation from R^m to R^n, then Im(T) is a linear subspace of R^n and Ker(T) is a linear subspace of R^m.

- **Theorem**: Let T be a linear transformation. Suppose that $T(X_0) = Y_0$. Then

- $\{X : T(X) = Y_0\} = \{X_0 + X' : X' \in Ker(T)\}$.

- Proof: $T(X) = Y_0$ iff
  - $T(X) = T(X_0)$ iff
  - $T(X - X_0) = 0$ iff
  - $X' := X - X_0 \in \mathrm{Ker}(T)$ iff
  - $X = X_0 + X'$ with $X' \in \mathrm{Ker}(T)$.