The inverse of a Linear Transformation

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Recall:

- Let S and T be sets.
- A mapping or function from S to T is a rule which assigns to every element s of S a (well-defined) element f(s) of T.
- The set S is called the source or domain of f. The set T is called the target or codomain of f.
- A function $f: S \to T$ is said to be invertible if for **every** t in T there is a **unique** s in S such that $f(s) = t$.

A closer look at f(s) =t

- f is said to be injective (one-to-one) if for every t there is **at most one** s such that f(s) = t.
- f is said to be surjective (onto) if for every t there is **at least one** s such that f(s) = t.
- f is said to be bijective if for every t there is **exactly one** s such that f(s) = t.

Inverse functions

Theorem: Let $f: S \to T$ be a function. Then the following conditions are equivalent.

- f is surjective and injective.
- f is bijective.
- There exists a function $g: T \to S$ such that
  - $g(f(s)) = s$ for all s and
  - $f(g(t)) = t$ for all t.
- This g is called the <u>inverse</u> of the function f.

The inverse of a linear transformation

- **Theorem**: Let A be an n x m matrix. Then $T_A: R^m \to R^n$ is invertible if and only if n = m = rank(A). If this is the case, its inverse T_A^{-1} is also linear.

Proof: Let r be the rank of A.

- If $r < m$, then there are $m-r$ free variables.
- Hence the equation $T_A(x) = 0$ has infinitely many solutions and T_A could not be injective.
- Thus if T_A is injective $m \leq r$, hence $m = r$.
- Conversely: if $m = r$, T_A is injective.

- If $r < n$, then there is some y such that the equation $T_A(x) = y$ is inconsistent. Hence T_A will not be surjective.
- Hence if T_A is surjective $n \leq r$, hence $n = r$.
- Conversely, if $n=r$, T_A is surjective.
- Thus if T_A is bijective, $n = r = m$.

Conclusion: If A is an n x m matrix of rank r,

1. T_A is injective if and only if m = r.
2. T_A is surjective if and only if n = r.

Corollary: If n = m, then the following are equivalent:

1. T_A is injective (i.e., r = m).
2. T_A is surjective (i.e., r = n).
3. T_A is bijective.

Computing T_A^{-1}

- If T_A is invertible, then, $(T_A)^{-1} = T_{(A^{-1})}$, another linear transformation, where A^{-1} is computed as follows:
  - Form the matrix $(A|I_n)$.
  - Put $(A|I_n)$ in reduced row echelon form. Then $\text{rref}(A|I_n) = (I_n|A^{-1})$.
  - See syllabus web page for a proof of this fact.