Linear Differential Equations

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Notation

We work over an open interval:
$I := (a, b) := \{t \in \mathbf{R} : a < t < b\}$.

Let $C^k(I) := \{f \colon I \to \mathbf{R} : f, f' \cdots f^{(k)}$ exist and are continuous $\}$.

Note that $C^k(I)$ is a linear subspace of the vector space of all functions from I to $\mathbf{R}$. We could also consider complex-valued functions, which form a vector space over the field of complex numbers.

The main existence and uniqueness theorem

Theorem

Given $p_1, p_2, \dots p_n \in C^0(I)$ *and* $f \in C^n(I)$*, let*

$$L(f) := f^{(n)} + p_1 f^{(n-1)} + \cdots + p_n f.$$

Then:

- L *is a linear transformation:* $C^n(I) \to C^0(I)$.
- *Given* $g \in C^0(I)$, $t_0 \in I$*, and* $y_0, y_1, \dots y_{n-1} \in \mathbf{R}$*, there exists a unique* $f \in C^n(I)$ *such that*
  - $L(f) = g$*, and*
  - $f^{(i)}(t_0) = y_i$ *for* $i = 0, 1, \dots, n-1$.

Consequences

Recall that $Ker(L) := \{f : L(f) = 0\}$. That is, $Ker(L)$ is the set of solutions to the homgeneous equation

$$f^{(n)} + p_1 f^{(n-1)} + \cdots + p_n f = 0.$$

Corollary

$Ker(L)$ is a linear subspace of the vector space $C^n(I)$.

Corollary

For each $t_0 \in I$,

$$E_{t_0} \colon Ker(L) \to \mathbf{R}^n$$

$$f \mapsto \left(f(t_0), f'(t_0), \ldots, f^{(n-1)}(t_0)\right)$$

is an isomorphism.

Corollary

The operator $L\colon C^n(I) \to C^0(I)$ *is surjective, and its kernel is a linear subspace of* $C^n(I)$ *of dimension n.*

Corollary

If $(f_1, \cdots f_n)$ *is a linearly independent sequence in the kernel* $Ker(L)$ *of L then* $(f_1, \ldots, f_n)$ *forms a basis for* $Ker(L)$*.*

The Wronskian

Corollary

Let $(f_1, \ldots, f_n)$ be a sequence of elements of $Ker(L)$, and let

$$W(f_1, \ldots, f_n) := \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}.$$

Then the following conditions are equivalent:

- *For some $t_0 \in I$, $W(f_1, \ldots, f_n)(t_0) \neq 0$.*
- *For some $t_0 \in I$, the sequence of vectors*

  $$E_{t_0}(f_1), E_{t_0}(f_2), \ldots, E_{t_0}(f_n)$$

  in $\mathbf{R}^n$ is linearly independent.
- *$(f_1, \ldots, f_n)$ is a basis for $Ker(L)$.*
- *$W(f_1, \ldots, f_n)(t) \neq 0$ for all $t \in I$.*

Abel's theorem

Theorem

If $(f_1, \ldots, f_n)$ *is a sequence of elements of* $Ker(L)$ *and* $W := W(f_1, \ldots, f_n)$*, then* $W' + pW = 0$*. Hence* $W = ce^{-P}$*, where* c *is some constant and* $P' = p_1$*.*

Note that this proves again the fact that W is either always zero or never zero.

Constant Coefficients

If $p_1, \ldots, p_n$ are constant and $g = 0$, a fundamental solution set to the equation can be found fairly easily. One method is to try exponential solutions of the form e^{rt}.

Theorem

Suppose r is a root of the polynomial

$$X^n + p_1 X^{n-1} + \cdots p_n = 0.$$

Then e^{rt} belongs to the kernel of L. If r has multiplicity m, then the functions

$$e^{rt}, te^{rt}, \ldots, t^{m-1} e^{rt}$$

all belong the the nullspace, and form a linearly independent sequence.

Example

Consider the equation $y''' - 2y'' + y' = 0$. The corresponding characeristic equation is $r^3 - 2r^2 + r = 0$.
This factors:

$$r^3 - 2r^+ r = r(r2 - 2r + 1) = r(r-1)^2$$

So a basis for the solution space should be (e^0, e^t, te^t).
Let's compute the Wronskian:

$$\begin{aligned} W &:= \begin{vmatrix} 1 & e^t & te^t \\ 0 & e^t & e^t + te^t \\ 0 & e^t & 2e^t + te^t \end{vmatrix} \\ &= \begin{vmatrix} 1 & e^t & te^t \\ 0 & e^t & e^t + te^t \\ 0 & 0 & e^t \end{vmatrix} \\ &= e^{2t} \end{aligned}$$

Note that $W' - 2W = 0$.

Another example

Recall the damped spring equation of the form

$$y'' + py' + qy = 0.$$

The characteristic equation is $r^2 + pr + q = 0$, which has roots:

$$r = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

For example, if $p = 2$ and $q = 5$, we get

$$r = -1 \pm 2i,$$

which leads to solutions $(e^{-t} sin2t, e^{-t} cos2t)$.