Gauss-Jordan Elimination

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- **Goal:** Given a system $\mathcal{E}$ of linear equations, find the solution set $S(\mathcal{E})$.
- **Method:** Find an *e*quivalent system $\mathcal{E}'$ which is easy to analyze. Since $\mathcal{E}$ and $\mathcal{E}'$ are equivalent, $S(\mathcal{E}) = S(\mathcal{E}')$. Find $\mathcal{E}'$ by combining equations to *eliminate* variables.
- **Technique:** Streamline the computation by writing the equations in matrix form, in which rows correspond to equations. Operations on the equations correspond to *row operations* on the matrix. The goal is to find $\mathcal{E}'$ in *reduced row echelon form*.

Definitions

- A *matrix* is a rectangular array of numbers.
- An $m \times n$ matrix has m rows and n columns.
- If A is a matrix, the ijth entry a_{ij} of A is the entry in the ith row and jth column.
- A *row vector* is a matrix with just one row.
- A *column vector* is a matrix with just one column.
- Sometimes we just say "vector" when we mean "column vector."

A system of m linear equations in n unknowns can be abbreviated by an $m \times (n + 1)$ (augmented) matrix.

Reduced row echelon form

- A matrix is said to be *zero* if all its entries are zero.
- If $R = (a_1, \ldots, a_n)$ is a nonzero row vector, its *leading entry* is its first nonzero entry, and its *leading index* ℓ is the place where this leading entry occurs. Thus $a_\ell \neq 0$ and $a_i = 0$ for $i < \ell$.
- For example, the leading entry of $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \end{pmatrix}$ is 1 and its leading index is 2.

Definition

A matrix A is in *reduced row echelon form* if it satisfies the following conditions:

1. All the nonzero rows are above all the zero rows.
2. The leading entry of every nonzero row lies to the left of the leading entries of the nonzero rows below it.
3. Every leading entry of every nonzero row is 1.
4. Every leading entry is the only nonzero entry in the column containing it.

A description in index notation.

Definition

An $m \times n$ matrix A is in *reduced row echelon form* if it satisfies the following conditions:

1. Let r be the number of nonzero rows of A (so that $0 \leq r \leq m$). Then the first r rows of A are nonzero and the remaining rows are zero.
2. Let ℓ_i be the leading index of the ith row. Then $\ell_1 < \ell_2 \cdots < \ell_r$.
3. For $1 \leq i \leq r$, $a_{i\ell_i} = 1$.
4. For $1 \leq i \leq r$ and $1 \leq i' \leq m$ and $i' \neq i$, $a_{i',\ell_i} = 0$.

The solution set of a system in reduced row echelon form.

Let $\mathcal{E}$ be a system of m equations in n unknowns. Suppose the corresponding $m \times (n+1)$ matrix is in reduced row echelon form. We can describe the solution set as follows.

- If the leading entry of the last nonzero row of A is in the last column, the equations are inconsistent and there are no solutions.
- The variables $(x_{\ell_1}, x_{\ell_2}, \cdots, x_{\ell_r})$ corresponding to the leading index columns are *fixed*, and the remaining variables are *free*.
- More precisely, for $1 \leq i \leq r$, the ith row of A corresponds to an equation which expresses x_{ℓ_i} in terms of the free variables.
- Thus, if the equations are not inconsistent, there are r fixed variables and $n - r$ free variables. Thus the solution set has $n - r$ "degrees of freedom."

Row operations

Theorem

Let A be an $m \times n$ matrix.

- *There is sequence of elementary row operations which transform A into a matrix A' in reduced row echelon form.*
- *The systems of equations $\mathcal{E}$ and $\mathcal{E}'$ corresponding to A and A' are equivalent, i.e, they have the same solution set.*

There are three kinds of elementary row operations:

- Add a multiple of some row to some *other* row.
- Multiply a row by some nonzero number.
- Interchange two rows.

Notice that each of these operations can be undone by an operation of the same type. Verify that the corresponding operation on systems of equations does not alter the solution set.

Summary

To solve a system $\mathcal{E}$ of m equations in n unknowns:

- Write the corresponding $m \times (n+1)$-matrix A.
- Perform a sequence of row operations to obtain a matrix A' in reduced row echelon form.
- Let $\mathcal{E}'$ be the system of equations corresponding to A'. Then $\mathcal{E}'$ and $\mathcal{E}$ have the same solution set, and $\mathcal{E}'$ can be easily described.

Remark

In fact there is a well-defined *algorithm* that a computer (or you) can use to go from a matrix A to a matrix in reduced row echelon form. Thus we can write $rref(A)$, meaning that $rref(A)$ is a well-defined *function* of A. But in fact more is true; any sequence of elementary row operation which yields a matrix in reduced row echelon form when applied to A will give the same answer. More precisely:

Theorem

Let A, A', and A'' be matrices. Assume that A' and A'' are in reduced row echelon form and that both A' and A'' can be obtained from A by a sequence of elementary row operations. Then $A' = A''$.

pause The proof of this is theorem is not in the book. I may have time to explain it later.