Fourier Series: Summary

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Fix $L > 0$ and let $I := [-L, L]$, that is, the set of real numbers x such that $-L \leq x \leq L$.

- Let $\mathcal{P}$ be the set of piecewise continuous fuctions from I to $\mathbf{R}$ (a linear subspace of the vector space of all such functions).
- For $f, g \in \mathcal{P}$, let $(f|g) := \frac{1}{L}\int_{-L}^{L} fg$.
- $u_n(x) := \cos(\frac{n\pi x}{L})$ for $n = 0, 1, 2, \ldots$.
- $v_n(x) := \sin(\frac{n\pi x}{L})$ for $n = 1, 2, 3 \ldots$.
- $W_N := span\{u_0, u_1, \ldots u_N, v_1, v_2, \ldots v_N\}$.

Theorem: W_N is a linear subspace of $\mathcal{P}$ of dimension $2N+1$, and $(u_0, \ldots u_N, v_0, \ldots v_N)$ is an orthogonal basis for W_N. Moreover

$$||u_n||^2 = ||v_n||^2 = 1 \text{ for } n > 0, \text{ and } ||u_0||^2 = 2.$$

Corollary: If $f \in \mathcal{P}$, let

$$a_n := (f|u_n) = \frac{1}{L}\int_{-L}^{L} f(x)u_n(x)dx$$

$$b_n := (f|v_n) = \frac{1}{L}\int_{-L}^{L} f(x)v_n(x)dx$$

$$\pi_N(f) := \frac{a_0}{2}u_0 + a_1u_1 + \cdots a_Nu_N + b_1v_1 + \cdots b_Nv_N.$$

Then $\pi_N(f)$ is the orthogonal projection of f onto W_N.

Odd and Even functions

Definition:

- $f\colon I \to \mathbf{R}$ is *even* if $f(-x) = f(x)$ for all x.
- $f\colon I \to \mathbf{R}$ is *odd* if $f(-x) = -f(x)$ for all x.

Let $\mathcal{P}_{ev}$ be the set of all even elements of $\mathcal{P}$, and let $\mathcal{P}_{odd}$ be the set of odd elements of $\mathcal{P}$.

Proposition:

1. $\mathcal{P}_{ev}$ and $\mathcal{P}_{odd}$ are linear subspaces of $\mathcal{P}$.
2. $\mathcal{P}_{ev} \cap \mathcal{P}_{odd} = \{\mathbf{0}\}$. Any $f \in \mathcal{P}$ can be written uniquely as a sum $f = f_{ev} + f_{odd}$, where $f_{ev} \in \mathcal{P}_{ev}$ and $f_{odd} \in \mathcal{P}_{odd}$.
3. $\mathcal{P}_{ev} \perp \mathcal{P}_{odd}$.
4. If $f \in \mathcal{P}_{odd}$, $\int_{-L}^{L} f = 0$, and if $f \in \mathcal{P}_{ev}$, $\int_{-L}^{L} f = 2\int_0^L$.
5. A function $f\colon [0, L) \to \mathbf{R}$ can be extended uniquely as a periodic even function with period $2L$. Alternatively, it can be extended uniquely as a periodic odd function with period $2L$.
6. If $f \in \mathcal{P}_{odd}$, then $a_n = 0$ for all n and

$$b_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx$$

7. If $f \in \mathcal{P}_{ev}$, then $b_n = 0$ for all n and

$$a_n = \frac{2}{L}\int_0^L f(x) cos\frac{n\pi x}{L} dx$$

Theorem: If $f \in \mathcal{P}$, let $f_N := \pi_N(f)$, that is:

$$f_N = \frac{a_0}{2} + a_1 \cos(\frac{\pi x}{L}) + \cdots + a_N \cos(\frac{N\pi x}{L}) + b_1 \sin(\frac{\pi x}{L}) + \cdots + b_N \sin(\frac{N\pi x}{L}).$$

1. $\lim_{N\to\infty} ||f - f_N|| = 0$, where $||h|| := \sqrt{(h|h)}$ for any h.
2. If f and f' are piecewise continuous, then for $-L < x < L$,

$$\lim_{N\to\infty} f_N(x) = \frac{f(x^-) + f(x^+)}{2}.$$

3. If f and f' are piecewise continuous, then for $x = \pm L$,

$$\lim_{N\to\infty} f_N(x) = \frac{f(L^-) + f(-L^+)}{2}.$$