# Eigenvectors

**Theorem**: Let $(w_1, \ldots, w_m)$ be nonzero eigenvectors of $A$ corresponding to distinct eigenvalues. Then $(w_1, \ldots, w_m)$ is linearly independent.

Proof: We use induction on $m$. If $m = 1$ the statement is trivial, since $w_1$ is by assumption not zero, hence $(w_1)$ is linearly independent. Assume that the theorem is true for $m - 1$. To prove it for $m$, we have to show that if $\sum a_i w_i = 0$, then each $a_i = 0$. We know that each $w_i$ is an eigenvector with eigenvalue $\lambda_i$, so $Aw_i = \lambda_i w_i$. We have the following equations:

$$\begin{aligned} a_1 w_1 + a_2 w_2 + \cdots a_m w_m &= 0 \\ A(a_1 w_1 + a_2 w_2 + \cdots a_m w_m) &= A0 \\ a_1 A w_1 + a_2 A w_2 + \cdots a_m A w_m &= 0 \\ a_1 \lambda_1 w_1 + a_2 \lambda_2 w_2 + \cdots a_m \lambda_m w_m &= 0 \end{aligned}$$

On the other hand, if we multiply the top equation by $\lambda_m$, we get

$$a_1 \lambda_m w_1 + a_2 \lambda_m w_2 + \cdots a_m \lambda_m w_m = 0$$

Subtracting this equation from the last equation above, we get:

$$a_1(\lambda_1 - \lambda_m) w_1 + a_2(\lambda_2 - \lambda_m) w_2 + \cdots a_{m-1}(\lambda_{m-1} - \lambda_m) w_{m-1} = 0$$

But we already know that the sequence $(w_1, \ldots w_{m-1})$ is linearly independent, so each coefficient above must vanish: $a_i(\lambda_i - \lambda_m) = 0$ for $i = 1, \ldots m - 1$. Since $\lambda_i - \lambda_m \neq 0$, this implies that $a_i = 0$ for $i = 1 \ldots m - 1$. Now the first equation reduced to the statement that $a_m w_m = 0$, and since $w_m \neq 0$, this implies that $a_m = 0$ too.

Note that this gives another proof that an $n \times n$ matrix has at most $n$ eigenvalues.

**Theorem:** Let $\lambda_1, \ldots \lambda_r$ be the (distinct) eigenvalues of of $A$ and for each $i$, let $\beta_i$ be a basis for $Eig_{\lambda_i}(A)$. Then the union (concatenation) $\beta$ of all the $\beta_i$ is linearlry independent.