Eigenvalues

Definition: Let A be a matrix and let $\lambda \in \mathbf{R}$.

$E_\lambda := \{X : AX = \lambda X\}$ is a linear subspace of M_{n1}, called the λ-*eigenspace* of A.

Elements of E_λ are called the λ-*eigenvectors* of A.

A number λ is said to be an *eigenvalue* of A if there is some nonzero λ-eigenvector of A.

Theorem: $E_\lambda = NS(A - \lambda I) = NS(\lambda I - A)$.
Hence the following are equivalent:

- λ is an eigenvalue of A.
- $NS(\lambda I - A) \neq \{\mathbf{0}\}$.
- $\lambda I - A$ is singular.
- $\det(A - \lambda I) = 0$.

Theorem-Definition: The *characteristic polynomial* of A is the expression $f_A(t) := \det(A - tI)$. It is a polynomial of degree n, and in fact:

$$f_A(t) = (-t)^n + (-t)^{n-1}\mathrm{tr}(A) + \cdots + \det(A).$$

Corollary: The eigenvalues of A are the roots of its characteristic polynomial.

Theorem: Similar matrices have the same characteristic polynomial. That is, if S is an invertible matrix and $B = S^{-1}AS$, then $f_B(t) = f_A(t)$.