## How to diagonalize a matrix

Let $A$ be an $n \times n$ matrix.

1. Compute the *characteristic polynomial*

   $$f_A(x) := det(A - xA).$$

   This is a monic polynomial of degree $n$.

2. Find the roots $\lambda_1 \dots \lambda_r m$ of $f_A(X)$, together with their multiplictiies $m_1, \dots m_r$. There are at most $n$ roots so $r \leq n$. In fact $m_1 + \cdots m_r = n$, if you are willing to include complex roots if necessary. These roots are the *eigenvalues* of $A$.

3. For each $i$, find an ordered basis $\beta_i$ for the $Eig_{\lambda_i}(A) = NS(\lambda_i I - A)$, using Gauss elimination. Each $\beta_i$ will be a list of $d_i$ vectors, where $d_i$ is the dimension of $Eig_{\lambda_i}(A)$.

4. Assemble all the bases you constructed above into single list $\beta$ of vectors. There will be a total of $d_1 + d_2 + \cdots d_m$ elements in this list.

5. **Theorem**: The sequence $\beta$ is automatically linearly independent. The matrix $A$ is diagonalizable if and only if $d_1 + \cdots d_r = n$, and this is true if and only if $d_i = m_i$ for all $i$. If this is the case, $\beta$ is a basis for $\mathbf{R}^n$, and the matrix $S$ whose columns are the vectors in $\beta$ vectors satisfies $AS = SD$, with $D$ diagonal.

Note: Each $d_i \geq 1$, so if all the roots of $f_A(X)$ are distinct, then $m = n$, each $d_i = 1$, $\sum d_i = n$, and $A$ is automatically diagonalizable.