Determinants

Theorem: For each n, there exists a unique function $\det\colon M_{n\times n} \to \mathbf{R}$ with the following properties

1. $\det I = 1$
2. $\det(AB) = \det(A)\det(B)$
3. $\det A = 0$ if and only if A is singular.
4. $\det A$ is unchanged by an elementary row operation of the first type.
5. $\det A^T = \det A$.
6. For any i,
$$\det A = \sum_j (-1)^{i+j} a_{ij} M_{ij},$$
where M_{ij} is the determinant of the $n-1$ by $n-1$ matrix obtained by deleting the ith row and jth column from A.

Here is another characterization of the determinant:

Theorem: For each n, there exists a unique function $det\colon M_{n\times n} \to \mathbf{R}$ with the follwoing properties:

1. $\det I = 1$.
2. det is a linear function of each variable. That is, if all the variables v_i except for the jth is fixed, then the resulting function of v_j is linear in v_j.
3. $\det(v_1, \ldots v_n) = 0$ if there exist $i < j$ with $v_i = v_j$.