Systems with constant coefficients

November 14, 2007

Suppose the entries of A are constant. Then the solutions to the homogeneous equation

$$\mathbf{x}' = A\mathbf{x}$$

can often be found explicitly using eigenvectors and eigenvalues

Theorem

If $\mathbf{v}_i \in \mathbf{R}^n$ is an eigenvector of A with eigenvalue λ_i, then the vector-function

$$\mathbf{x}_i := e^{\lambda_i t}\mathbf{v}_i$$

is a solution to the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Proof

We just check:

$$\begin{aligned}
\mathbf{x}_i' &= (e^{\lambda_i t}\mathbf{v}_i)' \\
&= (e^{\lambda_i t})'\mathbf{v}_i \\
&= e^{\lambda_i t}\lambda_i\mathbf{v}_i \\
&= e^{\lambda_i t}\mathbf{A}\mathbf{v}_i \\
&= \mathbf{A}(e^{\lambda_i t}\mathbf{v}_i) \\
&= \mathbf{A}\mathbf{x}_i
\end{aligned}$$

Main conclusion

If $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then

$$(e^{\lambda_1 t}\mathbf{v}_1, \ldots e^{\lambda_n t}\mathbf{v}_n)$$

is a basis for the space of solutions to the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.