Boundary Value Problems

November 27, 2007

Review: Initial Value Problems

Recall that the existence and uniqueness theorem for second order linear equations says the following.

Theorem 1: Let p, q and be real numbers. Let

$$W := \{y : y'' + py' + qy = 0\}.$$

1. W is a linear subspace of the space of all functions, and has dimension 2.
2. For any t_0, the map

$$W \to \mathbf{R}^2 : y \mapsto \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix}$$

is an isomorphism.

Boundary value problems

Now choose t_0, t_1 with $t_1 > t_0$, and consider the boundary value mapping

$$B\colon W \to \mathbf{R}^2 : y \mapsto \begin{pmatrix} y(t_0) \\ y(t_1) \end{pmatrix}$$

This is again a linear transformation, but it *not* always an isomorphism.

Theorem 2: In the above situation, let $\theta := \sqrt{4q - p^2}$ and let $\ell := t_1 - t_0$. Then there are two possible cases:

1. B has rank two, and so is an isomorphism. This happens whenever $p^2 \geq 4q$ or whenever $4q > p^2$ and $\ell \neq n\pi/\theta$ for some integer n. In either of these cases, given any y_0, y_1, there is a unique solution $y \in W$ with $y(t_i) = y_i$.
2. B has rank one. This happens when $4q > p^2$ and ℓ is an integer n times π/θ. In this case, the set of (y_0, y_1) in $\mathbf{R}^2$ for which a solution exists is a one-dimensional linear subspace of $\mathbf{R}^2$, and the set of all y such that $y(t_0) = 0 = y(t_1)$ is a one-dimensional linear subspace of W, with basis $\sin(\theta t) = \sin(\frac{n\pi}{\ell} t)$.