Practice Final

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly—explanations (with complete sentences when appropriate) will help us understand what you are doing. Math 49 students taking the linear algebra portion should work problems 1–5; Math 49 students taking the differential equation should work problems 6–10.

1. Let $A := \begin{pmatrix} 1 & -2 & 1 & 1 & -2 \\ -1 & 2 & -1 & 0 & 1 \\ 1 & -2 & 1 & -1 & 0 \end{pmatrix}$.
   - (a) Find a matrix B which is in reduced row echelon form and which is row equivalent to A.
   - (b) Find a basis for the null space of A.
   - (c) Find a basis for the column space of A, from among the columns of A.
   - (d) Find all X such that $AX = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.
   - (e) Find at least one X such that $A^T AX = A^T B$, where $B = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

     Hint: you do not need to calculate $A^T A$ to do this.
2. Write the definition of each of the following concepts. Use complete sentences and be as precise as you can.
   - (a) The *inverse* of a matrix.
   - (b) A *linearly independent sequence* in a vector space V.
   - (c) The *dimension* of a vector space. State the theorems which makes this definition meaningful.
   - (d) The *orthogonal projection* of a vector in $\mathbf{R}^n$ onto a linear subspace W.
   - (e) An *eigenvector* of a linear operator $T\colon V \to V$.
3. Let V be the space of vectors in $\mathbf{R}^4$ such that $x_1 + x_2 + x_3 + x_4 = 0$. and let W be the set of vectors in V such that $x_1 = x_4$.
   - (a) Find an orthogonal basis (v_1, v_2, v_3) for V with $v_1 = (0, 1, -1, 0)$ and such that (v_1, v_2) is a basis for W.
   - (b) Find the orthogonal projection of $v := (2, 1, 3, -6)$ on W.
   - (c) Find the distance from v to W.

4. Suppose that A is a matrix with 4 rows and 8 columns, and suppose that the rows of A span a three dimensional subspace of $\mathbf{R}^8$. Answer the following questions, explaining you reasoning.

   (a) What is the dimension of the space spanned by the columns of A?

   (b) What is the dimension of the null space of A?

   (c) What is the dimension of the null space of $A^T A$?

   (d) Prove that for any $m \times n$ real matrix A, the ranks of $A^T A$ and of A are the same.

5. Let $A = \begin{pmatrix} 6 & -1 \\ 4 & 2 \end{pmatrix}$.

   (a) Find the eigenvalues of A.

   (b) We know that there exist matrices S and T such that $A = STS^{-1}$, where T is upper triangular. Find T.

   (c) Now compute e^{tA}, as a function of t.

   (d) Find a matrix B with positive eigenvalues such that $B^2 = A$.

6. Consider the system of differential equations:

$$\begin{aligned} f' &= g \\ g' &= 2f - g \end{aligned}$$

   (a) Write this system as a vector-valued differential equation.

   (b) Find a fundamental solution set for the equation in part (a).

   (c) Compute the Wronskian of your solution set.

   (d) Find a pair of functions f, g satisfying the original system and such that $f(0) = -1$ and $g(0) = 5$.

7. For each of the following matrices, sketch and describe the trajectories of the solutions to the differential equation $Y'(t) = AY(t)$. In particular, exhibit any asymptotes and/or invariant lines, draw arrows indicating the direction of the flow along the solution, etc.

   (a) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$

   (d) $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

8. Suppose $Y''(x) = kY(x)$ and $Y(0) = 0$, $Y'(3) = 0$.

   (a) For which values of $k \in \mathbf{R}$ is there a nontrivial solution to this equation?

   (b) Explain (prove) why you have found all such k. (This part will count more than the actual answer.)

   (c) For each such k, give the corresponding solutions.

9. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is even, periodic with period 2π, and satisfies $f(x) = \sin x$ for $x \in [0, \pi]$.

   (a) Draw a sketch of the graph of f, labeling your axes carefully.

   (b) Find a Fourier series which represents f. Explain how you find the coefficients. You may use one or more of the formulas at the end of the test to evaluate the coefficients if you like.

   (c) Does the series converge to f at every point? Explain.

10. Suppose that a bar of length π with thermal coefficient $\alpha^2 = 2$ is insulated on its surface except at the end points.

    (a) Write the differential equation governing heat diffusion in the bar described above.

    (b) If the ends of the bar are kept at 0^o and the initial temperature distribution is $u(x, 0) = \sin(5x)$, find a formula for the temperature distribution at all times.

    (c) If instead one end is kept at π° and the other at $3\pi^\circ$, what is the limiting temperature distribution (steady state solution) of the temperature as $t \to \infty$? Verify directly that this limit distribution satisfies the equation in part (a).

    (d) In the situation (c), assume again that the initial temperature is given by $u(x, 0) = \sin(5x)$. Find a formula for the temperature distribution at time t. You may use one or more of the formulas listed at the end of the test.

Formulas

$$
\begin{aligned}
1 &= 2\sum \frac{1+(-1)^{k+1}}{\pi k}\sin(kx) \qquad \text{for } 0 \le x \le \pi \\
x &= 2\sum \frac{(-1)^{k+1}}{k}\sin(kx) \qquad \text{for } 0 \le x \le \pi
\end{aligned}
$$

$$
\begin{aligned}
\sin(nx)\cos(x) &= \frac{\sin((1+n)x) + \sin((n-1)x)}{2} \\
\cos(nx)\sin(x) &= \frac{\sin((1+n)x) + \sin((1-n)x)}{2}
\end{aligned}
$$