The trace map for projective space

For simplicity we work over an affine scheme $S = \operatorname{Spec} R$. Recall that if E is an R-module, then $\mathbf{P}E$ is the scheme $\operatorname{Proj} S^{\cdot} E$. On $\mathbf{P}E$ there is a canonical exact sequence of quasi-coherent sheaves:

$$0 \to \mathcal{H} \to E \otimes \mathcal{O}_{\mathbf{P}E} \to \mathcal{O}_{\mathbf{P}E}(1) \to 0.$$

Here we have written $E \otimes \mathcal{O}_{\mathbf{P}E}$ to mean $\pi^* \tilde{E}$, where $\tilde{E}$ is the quasi-coherent sheaf on S associated to the R-module E. The map $u \colon E \otimes \mathcal{O}_{\mathbf{P}E} \to \mathcal{O}_{\mathbf{P}E}(1)$ is the universal invertible quotient of E, and $\mathcal{H}$ is the universal hyperplane in E. Tensoring the above sequence with $\mathcal{O}_{\mathbf{P}E}(-1)$, we find:

$$0 \to \mathcal{H}(-1) \to E \otimes \mathcal{O}_{\mathbf{P}E}(-1) \to \mathcal{O}_{\mathbf{P}E} \to 0.$$

We shall use the fact that there is a canonical isomorphism

$$\mathcal{H}(-1) \cong \Omega^1_{\mathbf{P}E/R},$$

Now suppose that E is projective of rank $n+1$. Then the isomorphism above induces an isomorphism:

$$\Lambda^{n+1}\mathcal{E}(-n-1) \cong \Omega^n_{\mathbf{P}E/S}.$$

The Koszul complex of the homomorphism $E \otimes \mathcal{O}_{\mathbf{P}E}(-1) \to \mathcal{O}_{\mathbf{P}E} \to 0$ is the complex:

$$0 \to \Lambda^{n+1}\mathcal{E}(-n-1) \to \Lambda^n \mathcal{E}(-n) \to \cdots \to \mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}E} \to 0.$$

It is exact becuase the mapping u is surjective. Thus the complex:

$$K^{\cdot} := \Lambda^n \mathcal{E}(-n) \to \cdots \to \mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}E}$$

is an n-term resolution of the complex $\Lambda^{n+1}\mathcal{E}(-n-1) \cong \Omega^n_{\mathbf{P}E/R}$. By the projection formula,

$$H^q(\mathbf{P}E, K^j) = \Lambda^{n-j}E \otimes H^q(\mathbf{P}E, \mathcal{O}_{\mathbf{P}E}(j-n)).$$

Since $j - n > -n-1$, these groups vanish if $q > 0$, so the complex $K^{\cdot}$ is in fact an *acyclic* resolution of $\Omega^n_{\mathbf{P}E/R}$. Hence the complex of global sections $H^0(\mathbf{P}E, K^{\cdot})$ calculates the cohomology of $\Omega^n_{\mathbf{P}E/R}$. But this complex vanishes except in degree n, where it is canonically isomorphic to R. We deduce a canonical (and coordinate free) isomorphism

$$H^n(\mathbf{P}E, \Omega^n_{\mathbf{P}E/R}) \cong R$$