Differentials and Smoothness

February 17, 2017

We present a supplement, and in some cases alternative, to Hartshorne’s Chapter II.§8 and Chapter II.§10. These notes are not meant to be self-contained, and will required reading Hartshorne for some definitions, some statements, and many important examples. We often do not include detailed proofs.

1 Derivations and deformations

Definition 1 Let $n$ be a natural number. An “$n$th order thickening” is a closed immersion $i: S \to T$ which is defined by an ideal $\mathcal{I}$ such that $\mathcal{I}^{n+1} = 0$. A “nilpotent thickening” is a closed immersion which is an $n$th order thickening for some $n > 0$.

If $Y$ is a scheme, a thickening in the category of $Y$-schemes means a thickening $i: S \to T$ of $Y$-schemes. Sometimes to emphasize this we may want to write $i: S/Y \to T/Y$.

The underlying map of topological spaces of a nilpotent thickening is a homeomorphism, and we will sometimes identify the underlying spaces of $S$ and $T$. First order thickenings are especially convenient to work with. If $\mathcal{I}$ is the ideal of a first order thickening $i: S \to T$, then $\mathcal{I}^2 = 0$, and this means that if $a$ is a local section of $\mathcal{O}_T$ and $x$ of $\mathcal{I}$, then the product $ax$ depends only on the image of $a$ in $\mathcal{O}_S$. Thus $\mathcal{I}$ has a natural structure of an $\mathcal{O}_S$-module: the natural map $\mathcal{I} \to i^* i^*(\mathcal{I})$ is an isomorphism. We will sometimes identify $\mathcal{I}$ with $i^*(\mathcal{I})$.

Example 2 If $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-modules, there is a scheme $D(\mathcal{M})$ whose underlying topological space is $S$ and whose structure sheaf
is $O_S \oplus M$, with the obvious $O_S$-module structure and with multiplication defined by $(a, x)(b, y) := (ab, ay + bx)$. Then the map $(a, x) \to a$ defines a first order thickening $i: S \to D(M)$ whose ideal sheaf $I$ identifies with $M$. The map $O_S \to O_S \oplus M$ sending $a$ to $(a, 0)$, defines a morphism $\rho: D(M) \to S$ such that $\rho \circ i = \text{id}$.

**Definition 3** Let $F$ be a presheaf on the category of $Y$-schemes, let $i: S \to T$ be a nilpotent thickening, and let $\xi$ be an element of $F(S)$. Then a “deformation of $\xi$ to $T$” is an element $\zeta$ of $F(T)$ such that $F(i)(\zeta) = \xi$, and we write $\text{Def}_\xi(T)$ to denote the set of all such elements.

If $T' \to T$ is any morphism and $S' := S \times_T T'$, then $S' \to T'$ is an $n$th order thickening, and a deformation of $\xi$ to $T$ pulls back to a deformation of the pullback $\xi'$ to $S'$ to $T'$. Thus $\text{Def}_\xi$ becomes a presheaf on the category of $T$-schemes. If $F$ is a sheaf, then $\text{Def}_\xi$ defines a sheaf on the Zariski topology of $T$, equivalently on the Zariski topology of $S$.

If a thickening $S \to T$ admits a retraction $\rho: T \to S$ (as in Example 2), then $F(\rho)(\xi)$ is automatically a deformation of $\xi$.

**Definition 4** Let $f: X \to Y$ be a morphism of schemes and let $E$ be a sheaf of $O_X$-modules. Then $\text{Der}_{X/Y}(E)$ is the sheaf of $f^{-1}(O_Y)$-linear maps $D: O_X \to E$ such that $D(ab) = D(a)b + aD(b)$ for any two local sections $a, b$ of $O_X$.

Note that $\text{Der}_{X/Y}$ is an $O_X$-submodule of $\text{Hom}(O_X, E)$.

The following notion is extremely pervasive in mathematics.

**Definition 5** Let $X$ be a topological space, let $G$ be a sheaf of groups on $X$. A (left) $G$-pseudo torsor is a sheaf of (left) $G$-sets: $G \times S \to S$ such that the corresponding map

$$G \times S \to S \times S: (g, s) \mapsto (gs, s)$$

is an isomorphism. A torsor is a pseudo-torsor each of whose stalks is nonempty.

There is an obvious notion of a morphism of $G$-torsors, and thus an obvious notion of the category of $G$-torsors on $X$ and of the set of isomorphism classes of $G$-torsors. If $S$ is a $G$-torsor and $U$ is an open subset of $X$ and if $S(U)$ is not empty, then $S(U)$ is isomorphic to $G(U)$: given an $s \in S(U)$, the map $G(U) \to S(U) : g \mapsto gs$ is an isomorphism of left $G(U)$-sets.
Theorem 6 Let $f: X \to Y$ be a morphism of schemes and let $X/Y$ be the functor $h_X$ from the category of $Y$-schemes to the category of sets. Let $i: S \to T$ be a first-order thickening, defined by an ideal $\mathcal{I}$, and let $g: S \to X$ be an element of $X/Y(S)$. Then the sheaf $g_*(\text{Def}_g)$ has a natural structure (see the formula below) of a pseudo-torsor under the sheaf of groups

$$\text{Der}_{X/Y}(\mathcal{O}_X, g_*\mathcal{I}) \cong \text{Hom}(\Omega^1_{X/Y}, g_*\mathcal{I}).$$

Proof: A deformation $h$ of $g$ is a morphism $h: T \to X$ such that $h \circ i = g$. Since $i$ is a homeomorphism and $g$ is given, to give $h$ is the same as to give a homomorphism $\tilde{h}: \mathcal{O}_X \to g_*\mathcal{O}_T$. Let $D: \mathcal{O}_X \to g_*\mathcal{I}$ be a derivation, and define $\tilde{h}: \mathcal{O}_X \to g_*\mathcal{O}_T$ to be $D + \tilde{h}$. This map is $\mathcal{O}_Y$-linear, and we claim that in fact it is a homomorphism. If $a, b \in \mathcal{O}_X$,

$$(D + \tilde{h})(ab) = D(ab) + \tilde{h}(ab) = aDb + bDa + \tilde{h}(a)\tilde{h}(b),$$

while

$$(D + \tilde{h})(a)(D + \tilde{h}(b)) = D(a)D(b) + \tilde{h}(b)D(a)\tilde{h}(a)D(b) + \tilde{h}(a)\tilde{h}(b).$$

Since $\mathcal{I}^2 = 0$, $D(a)D(b) = 0$, and $\tilde{h}(b)D(a) = g^\sharp(b)D(a) = bD(a)$; and similarly $\tilde{h}(a)D(b) = g^\sharp(a)D(b) = aD(b)$, and $D$ really is a derivation.

On the other hand, if $h_1$ and $h_2$ are deformations of $g$, then the $\mathcal{O}_Y$-linear map $h_2^\sharp - h_1^\sharp$ factors through $\mathcal{I}$, and it is easy to check that this difference is a derivation $\mathcal{O}_X \to g_*\mathcal{I}$.

For example if $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, we have a standard deformation $\rho \in \text{Def}_{id}(D(\mathcal{M}))$, and if $D \in \text{Der}_{X/Y}(\mathcal{M})$, then $D + \rho$ is the deformation given by $a \mapsto (a, Da) \in \mathcal{O}_X \oplus \mathcal{M}$.

Corollary 7 Let $f: X \to Y$ be a morphism of schemes, and let $\mathcal{I}_{X/Y}$ be the ideal of the (locally closed) diagonal embedding: $\Delta: X \to X \times_Y X$. Then the map

$$d: \mathcal{O}_X \to \mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2: a \mapsto p_2^\sharp(a) - p_1^\sharp(a)$$

is a universal derivation.

Proof: Let $T$ be the locally closed subscheme of $X \times_Y X$ defined by $\mathcal{I}_{X/Y}^2$. Then $X \to T$ is a first order thickening, and $p_2$ and $p_1$ are two deformations of $id X$. It follows that their difference $d$ is a derivation. Suppose $D: \mathcal{O}_X \to$
\[ \mathcal{M} \text{ is any derivation of } X/Y; \text{ we claim that } D \text{ factors through a unique } O_X\text{-linear map } I_{X/Y}/I_{X/Y}^2 \to \mathcal{M}. \] We already know that the target of the universal derivation is quasi-coherent, so we may assume that \( \mathcal{M} \) is also quasi-coherent. Then the homomorphism \( O_X \to O_X \oplus \mathcal{M} : a \mapsto (a, Da) \) defines a morphism of schemes \( \xi : D(M) \to X \), and the pair \( (\rho, \xi) \) defines a morphism \( D(M) \to X \times_Y X \). These morphisms agree on \( X \subseteq D(M) \) and hence sends \( I_{X/Y} \) to the ideal \( I = (0, \mathcal{M}) \) of \( X \) in \( D(M) \). Since \( I^2 = 0 \), this morphism induces a map \( h : I_{X/Y}/I_{X/Y}^2 \to I \). Moreover, if \( a \in O_X \),

\[
h(p_2^*(a) - p_1^*(a)) = \xi^*(a) - \rho^*(a) = D(a).\]

To check the uniqueness of \( h \), let us observe that \( p_1^* I_{X/Y}/I_{X/Y}^2 \) is generated as a sheaf of \( O_X \)-modules the image of \( d \). Working locally, we may assume that \( X/Y \) is given by a homomorphism \( R \to A \). Then \( I_{X/Y} \) is the sheaf associated with the ideal \( K := \text{Ker}(A \otimes_R A \to A) \). Say \( \sum a_i \otimes b_i \in K \). Then \( \sum a_i b_i = 0 \), so

\[
\sum a_i \otimes b_i = \sum a_i \otimes b_i - \sum a_i b_i \otimes 1 = \sum a_i (1 \otimes b_i - b_i \otimes 1).\]

\[ \square \]

The exact sequences in the following result help with the calculation of sheaves of differentials. The proof is an easy consequence of the universal mapping property of differentials.

**Theorem 8** Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of schemes and let \( h := g \circ f \).

1. The natural maps fit into an exact sequence:

\[
g^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0
\]

2. if \( f \) is a closed immersion defined by an ideal sheaf \( \mathcal{I} \), \( \Omega_{X/Y} = 0 \), and there is an exact sequence:

\[
I/I^2 \xrightarrow{\delta} f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to 0
\]

where \( \delta \) fits into a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{I} & \xrightarrow{d} & \Omega_{Y/Z} \\
\downarrow & & \\
I/I^2 & \xrightarrow{\delta} & f^* \Omega_{Y/Z}
\end{array}
\]
3. If $b:Y' \to Y$ is a morphism and $X' := Y' \times_Y X$, the natural map 
$pr^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$, is an isomorphism.

**Example 9** In the situation of part (2) of Theorem 8, suppose that $X = Z$ and that $h = \text{id}_X$. Then the sequence (2) induces an isomorphism 
$I/I^2 \to f^*\Omega_{Y/Z}$.

Indeed, it is clear that $\Omega_{X/X} = 0$, and it remains only to prove that the map 
$\delta: I/I^2 \to i^*\Omega_{X/Y}$
is injective. We will construct an inverse of this map as follows. Let $Y_1$ be the closed subscheme of $Y$ defined by $I^2$, and let $g_1 = g \circ h: Y \to X$. Note that $f_1$ is a first-order thickening in the category of $X$-schemes, since $g_1 \circ f_1 = \text{id}_X$. Let $h := f \circ g \circ h: Y_1 \to Y$, and note that $h \circ f = f$. That is, $h$ and $\tilde{h}$ are two deformations of $f$ to $Y_1$, and hence $h^* - \tilde{h}^*$ is a derivation $O_Y \to I/I^2$. If $a \in I$, then 
$$D(a) = h^*(a) - \tilde{h}^*(a) = h^*(a) - h^*g^*(f^*(a)) = h^*(a)$$
Thus $D$ defines a homomorphism $\Omega^1_{Y/X}$ sending $da$ to the class of $a$ in $I/I^2$ and provides a splitting of the map $\delta$.

For example, suppose that $Y$ is a scheme of finite type over a field $k$ and the inclusion $f: X \to Y$ is corresponds to a $k$-rational point $x$ of $Y$. Then we get an isomorphism: 
$$m_x/m_x^2 \cong \Omega_{Y/k}(x).$$
Thus in this case the Zariski tangent space of $Y$ at $x$, the $k$-dual of $m_x/m_x^2$, equivalently the set of deformations of the inclusion $x \to Y$ to the dual numbers $D_k(\epsilon)$, becomes identified with the set of maps $\Omega_{Y/k} \to i_*k(x)$, that is, with the fiber of $V(\Omega_{Y/k})$ over $x$. In general, $V\Omega_{Y/Z}$ is called the tangent space (or bundle) of $Y/Z$.

**Corollary 10** Let $X/k$ be a scheme locally of finite type, where $k$ is algebraically closed. Then the dimension of $m_x/m_x^2$ is an uppersemicontinuous function on the set of closed points of $X$.

**Proof:** This is because $\Omega_{X/k}$ is a quasi-coherent sheaf of finite type on $X$ (hence coherent, since $X$ is noetherian), and it follows from Nakayama’s lemma that the dimension of $\Omega_{X/k}(x)$ is upper semicontinuous. In fact: $\square$
Lemma 11 Let $X$ be a scheme and let $E$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules which is locally finitely generated. Then the dimension of $E(x)$ is uppersemicontinuous on $X$. If $E$ is locally free, it is in fact locally constant, and the converse holds if $X$ is reduced.

Proof: Let $x$ be a point of $X$, and let $(e_1(x), \ldots, e_d(x))$ be a basis for $E(x) := E_x/m_{X,x}E_x$. There exist an open affine neighborhood $U$ of $x$ and sections $(e_1, \ldots, e_d)$ of $E(U)$, such that the image of $e_i$ in $E(x)$ is $e_i(x)$. Replace $X$ by $U$ and let $(\mathcal{O}_X)^n \to E$ be the corresponding map. It follows from Nakayama’s lemma that the induced map on $(\mathcal{O}_{X,x})^n \to E_x$ is surjective, and hence that is is surjective in some neighborhood of $x$ (again using the fact that $E$ is finitely generated). Then $\dim E(x') \leq \dim E(x)$ for all $x'$ in this neighborhood. Suppose the dimension is in fact constant. We may assume it is constant and that $X$ is affine, say $X = \text{Spec } A$, and $E$ corresponds to a finitely generated $A$-module $M$. We have constructed a surjective map $A^n \to M$, where $n$ is the dimension of $M \otimes k(x)$ for every $x \in \text{Spec } A$. It follows that the map $k(x)^n \to M \otimes k(x)$ is bijective for every $x$. Let $K \subseteq A^n$ be the kernel of $A^n \to M$, and observe that any coordinate of any element of the kernel maps to zero in $A_p/PA_p$ for every prime $P$. Since $A$ is reduced, the intersection of all the primes is zero, so $K = 0$. \hfill \Box

Example 12 The map $I/I^2 \to i^*\Omega_{X/Y}$ might not be injective, even if $I$ is the maximal ideal corresponding to a closed point of a scheme $X$ of finite type over $Y = \text{Spec } k$. For example, let $k$ be a field of characteristic $p$ which is not perfect, with an element $a$ which is not a $p$th power, let $X := \text{Spec } k[X]$ and let $I$ be the ideal generated by $f := X^p - a$. Since this polynomial is irreducible, $I$ is maximal and corresponds to a closed point $x$. But $df = 0$, so the map in this case is zero.

2 Smooth, unramified, and étale morphisms

The following definition makes sense for any morphism of sheaves $X \to Y$, but we state it only for schemes.

Definition 13 A morphism $f: X \to Y$ is

1. formally smooth if for every affine $n$th order thickening $S \to T$ over $Y$ for every $g \in X/Y(S)$, can be deformed to $T$;
2. formally unramified if for every $S \to T$ every $g$ as above, has at most one deformation to $T$.

3. formally étale if it is both formally smooth and formally unramified.

A morphism is smooth if it is formally smooth and locally of finite presentation, is unramified if it is formally unramified and locally of finite type, and is étale if it is smooth and unramified.

If $f$ is formally étale, then the uniqueness of the deformations implies that the liftings on open covers agree on the overlaps and hence patch to a unique global lifting. Since every $n$th order thickening is a succession of first order thickenings, it suffice in the above definition just to consider first order thickenings. In this case, let $\mathcal{I}$ be the ideal sheaf of $S$ in $T$. Then $\text{Def}_g$ forms a pseudo-torsor under $\text{Der}_{X/Y}(g, \mathcal{I})$, and to say that $f$ is formally smooth is to say that this pseudo-torsor is always in fact a torsor. To say that $f$ is formally unramified is to say that the pseudo-torsor has at most one element.

**Corollary 14** A morphism $f: X \to Y$ is formally unramified if and only if $\Omega^1_{X/Y} = 0$.

**Proof:** If $\Omega^1_{X/Y} = 0$, it follows from Theorem 6 that first order deformations are unique if they exist. Suppose conversely that $f$ is unramified, let $\mathcal{E}$ be any quasi-coherent sheaf of $X$ and let $X \to D_X(\mathcal{E})$ be the trivial extension of $X$ by $\mathcal{E}$. Then the set of deformations of $\text{id}_X$ to $D_X(\mathcal{E})$ is not empty and is a torsor under $\text{Hom}(\Omega^1_{X/Y}, \mathcal{E})$. It follows that this group is zero. Taking $\mathcal{E} = \Omega^1_{X/Y}$, we see that the latter must vanish. \hfill $\square$

**Theorem 15** Let $f: X \to Y$ be a morphism of finite type. Then $f$ is unramified if and only if its geometric fibers are finite, reduced, and discrete.

**Proof:** Since $f$ is of finite type, the sheaf $\Omega^1_{X/Y}$ is also of finite type. It vanishes if and only if for each $x \in X$, the stalk $\Omega^1_{X/Y}$ at $x$ vanishes, and by Nakayama’s lemma, this is true if and only if the fiber $\Omega^1_{X/Y}(x)$ vanishes. Let $y := f(x)$, which we identify with $\text{Spec} \ k(y) \to Y$, and let $X_y := X \times_Y y$. Then if $p: X_y \to Y$ is the natural map, $p^*\Omega^1_{X/Y} \cong \Omega^1_{X_y/y}$. The point $x$ of $X$ defines a point $x'$ of $X_y$ with $p(x') = x$, and the above isomorphism identifies $\Omega^1_{X_y}(x')$ with $\Omega^1_{X/Y}(x)$. Thus we see that $f$ is unramified if and only if every $X_y \to y$ is unramified. Since $\overline{\gamma} \to y$ is faithfully flat, $\Omega^1_{X_y/y} = 0$ if and only if $\Omega^1_{X_y/\overline{\gamma}}$ is unramified. \hfill $\square$
Proposition 16  If \( f: X \to Y \) is smooth, then the sheaf \( \Omega^1_{X/Y} \) is locally free.

Theorem 17  Let \( f: X \to Y \) and \( g: Y \to Z \) be morphisms of schemes, each locally of finite presentation, and let \( h := g \circ f \).

1. Suppose that \( f: X \to Y \) is a closed immersion defined by a sheaf of ideals \( I \). If \( X \to Z \) is smooth, the map

\[
\bar{d}: I/I^2 \to f^*\Omega^1_{Y/Z}
\]

of Theorem 8 is injective and locally split. The converse is true provided that \( Y \to Z \) is smooth.

2. If \( f \) is smooth, the map

\[
f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z}
\]

of Theorem 8. is injective and locally split. The converse is true provided that \( h \) is smooth.

Proof: Suppose that \( h: X \to Z \) is smooth. Let \( j: Y_1 \to Y \) be the closed subscheme of \( Y \) defined by \( I^2 \). Then \( i: X \to Y_1 \) is a first-order thickening.

Assuming without loss of generality that \( X \) is affine, the smoothness of \( X/Z \) implies that there is a retraction \( r: Y_1 \to X \), compatible with the given maps to \( Z \). Let \( \bar{j} := j \circ r: Y_1 \to Y \), and note that \( \bar{j} \circ i = f \circ r \circ i = f = j \circ i \).

Thus \( j^* - \bar{j}^* \) “is” a derivation \( D: O_Y \to I/I^2 \). This derivation defines a map \( \Omega^1_{Y/Z} \to I/I^2 \) which gives the desired splitting.

For the converse, suppose that \( g \) is smooth, working locally, that

\[
s: f^*\Omega^1_{Y/Z} \to I/I^2
\]
splits \( \bar{d} \). Let \( i: S \to T \) be an affine first order thickening over \( Z \), defined by a square zero ideal \( J \), and let \( r: S \to X \) be a morphism (over \( Z \)). Since \( Y/Z \) is smooth, there exists a deformation \( s \) of \( f \) to \( T \).

Then \( s^2: O_Y \to s_* O_T \) necessarily maps \( I \) to \( r_* J \), and since \( J^2 = 0 \), factors through a map \( \theta: I/I^2 \to s_* J \). Then \( \theta \circ s: f^*\Omega^1_{Y/Z} \to J \), composed with the natural map \( \Omega^1_{Y/Z} \to f_* f^*\Omega^1_{Y/Z} \) defines a derivation \( D: O_Y \to s_* J \). Then \( \bar{s} := s - D \) is a map \( T \to Y \), and in fact \( \bar{s}^2 \) kills \( I \) and hence \( \bar{s} \) factors through \( X \).

We omit the proof of (2), which is quite similar.

\( \square \)
Remark 18 In the situation of part (1) of the previous theorem, let $x$ be a point of $X$, and suppose that $Y \to Z$ is smooth. The morphism $X \to Z$ is smooth in some neighborhood of $x$ if and only if the map of $k(x)$-vector spaces:

$$d(x): I(x) \to \Omega^1_{Y/Z}(x)$$

is injective. Smoothness implies that $d$ is injective and locally split, and it follows immediately that $d$ is injective. Conversely, if $d(x)$ is injective, choose a lift $f_1, \ldots, f_r$ to $I_x$ of a basis for $I(x)$. By Nakayama’s lemma, these elements generate $I_x$. (Recall that $X$ is of finite presentation.) Then $(df_1(x), \ldots, df_r(x))$ remains linearly independent in $\Omega^1_{Y/Z}(x)$, and hence can be completed to a basis $(df_1(x), \ldots, df_r(x), \omega_1(x), \omega_2(x))$, with $\omega_i \in \Omega^1_{Y/Z,x}$. Since $\Omega^1_{Y/Z,x}$ is free, it follows that $(df_1, \ldots df_r, \omega_1, \omega_2)$ is a basis, and the desired splitting is easy to construct.

Corollary 19 Let $X/k$ be a scheme of finite type over a field $k$ and let $x$ be a $k$-rational point of $X$. Then $X/k$ is smooth in some neighborhood of $x$ if and only if $X$ is regular at $x$.

Proof: The question is local, so we may assume that $X$ can be embedded in $Y := \mathbb{A}^n$. If $m_x \mathcal{O}_{Y,x}$ is the maximal ideal of the local ring $\mathcal{O}_{Y,x}$, the map $d(x)$ above identifies with the map $I_x/m_x I_x \to m_x/m_x^2$. Thus the map is injective if and only if $I_x \cap m_x^2 \subseteq m_x I_x$. Since $Y$ is regular at $x$, this condition is equivalent to the regularity of $X$ at $x$.

Let us recall the proof. Since the closed points of $X$ are dense and since localizations of regular local rings are regular, $X$ is a regular scheme if and only if the local rings at the closed points of $X$ are regular. Working locally, we may assume that $X$ can be embedded as a closed subscheme of an affine space $Y/k$. Let $x$ be a closed point of $X$. Corollary 18 tells us that $X$ is smooth if and only if the map $\mathcal{I}(x) \to m_x/m_x^2$ is injective. This is equivalent to the regularity of $X$ at $x$. Let us recall the proof. Let $m_x$ be the maximal ideal of $\mathcal{O}_{Y,x}$ and let $\overline{m}_x = m_x/I_x$ be the maximal ideal of $\mathcal{O}_{X,x}$. We have exact sequences:

$$0 \to \mathcal{I}_x \cap m_x^2/\mathcal{I}_x m_x \to \mathcal{I}(x) \to \mathcal{I}_x/\mathcal{I}_x \cap m_x^2 \to 0$$

$$0 \to \mathcal{I}_x/\mathcal{I}_x \cap m_x^2 \to m_x/m_x^2 \to \overline{m}_x/\overline{m}_x^2 \to 0$$

where $m_x$ is the maximal ideal of $\mathcal{O}_{Y,y}$ and $\overline{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. For simplicity of notation, rewrite these sequences as:

$$0 \to K(x) \to \mathcal{I}(x) \to \overline{\mathcal{I}(x)}.$$
0 → \mathcal{I}(x) → m_x/m_x^2 → \overline{m}_x/\overline{m}_x^2 → 0.

Since \( Y \) is regular, \( \dim_x(Y) = \dim m_x/m_x^2 \), and since \( \mathcal{I}_x \) can be generated by \( \dim \mathcal{I}(x) \) elements,

\[
\dim_x(X) \geq \dim_x(Y) - \dim \mathcal{I}(x) \\
\geq \dim(m_x/m_x^2) - \dim(K(x)) - \dim(\mathcal{I}(x)) \\
\geq \dim(\overline{m}_x/\overline{m}_x^2) - \dim(K(x))
\]

If \( K(x) = 0 \) we conclude that \( \dim_x(X) \geq \dim(\overline{m}_x/\overline{m}_x^2) \), and it follows that in fact equality holds and \( X \) is regular at \( x \). For the converse, suppose that \( X \) is regular and choose a sequence of elements \( f_1, \ldots, f_r \) of \( \mathcal{I}_x \) lifting a basis for \( \mathcal{I}(x) \). Then \( f_1(x) \cdots f_r(x) \in \mathcal{I}(x) \) are linearly independent in \( m_x/m_x^2 \), and hence can be completed to a basis for this vector space. Lifting the remaining elements to elements of \( m_x \), we end up with a sequence \( (f_1, \ldots, f_n) \) of elements of \( m_x \) such that \( (f_1, \ldots, f_r) \) lie in \( \mathcal{I}_x \) and such that \( f_1(x), \ldots, f_n(x) \) is a basis for the maximal \( m_x \) of \( x \) in \( \mathcal{O}_{Y,x} \). Let \( J \) be the ideal \( \mathcal{O}_{Y,y} \) generated by \( (f_1, \ldots, f_r) \). Then it follows from the argument above that \( X' := \text{Spec} \mathcal{O}_{Y,y}/J \) is regular of dimension \( \dim Y - \dim(\mathcal{I}_x) = \dim m_x/\overline{m}_x^2 \). If \( X \) is regular at \( x \), then this is also the dimension of \( X \). But \( X \subseteq X' \) and since \( X' \) is regular, it is irreducible, and it follows that \( X \) and \( X' \) coincide at \( x \). Then \( J = \mathcal{I}_x \) and it follows that \( K(x) = 0 \).

**Theorem 20** Let \( f: X \to Z \) be a morphism locally of finite presentation. Assume that \( X \) and \( Y \) are locally noetherian. Then \( f \) is smooth if and only if it is flat and its geometric fibers are regular.

**Proof:** Assume that \( f \) is smooth. Let \( x \) be a point of \( X \), let \( z := f(x) \), and let \( \overline{z} \) be the spectrum of algebraically closed field endowed with a map to \( z \). Then \( X_{\overline{z}} \) is smooth, and hence by Corollary 19 it is regular. Our task is to prove that \( f \) is flat. We use the following technique from commutative algebra.

**Lemma 21** Let \( R \to B \) be a local homomorphism of noetherian local rings.

1. \( B \) is flat over \( R \) if and only if \( \text{Tor}_1^R(B, k_R) = 0 \).

2. Suppose that \( B \) is flat over \( R \) and that \( A \) is the quotient of \( B \) by the ideal \( I \) generated by an element \( b \) of \( m_B \). If the image \( \overline{b} \) of \( b \) in \( B/m_RB \) is a nonzero divisor, then \( A \) is also flat over \( R \).
Proof: Statement (1) is the famous “local criterion of flatness,” and we use it to prove (2). Since $B$ is $R$ flat, $\text{Tor}_1^R(B,k_R) = 0$, so the top row of the diagram below is exact:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Tor}_1^R(A,k_R) & \rightarrow & I \otimes_R k_R & \beta \\
& & \downarrow{\pi} & & \downarrow{\bar{b}} \\
& & B \otimes_R k_R \\
\end{array}
\]

By assumption \( \bar{b} \) is injective, and it follows that \( \pi \) is also injective. The map \( \pi \) is obtained from the surjective map \( B \rightarrow I \) and hence is also surjective, hence bijective. Then it follows that \( \beta \) is injective and then that \( \text{Tor}_1^R(A,k_R) = 0 \).

We return to the proof of the theorem. Since \( X_\pi \rightarrow X_z \) is flat, it follows that \( X_z \) is regular, and in particular is an integral domain. Working locally, we may assume that there exists a closed immersion \( i: X \rightarrow Y \), where \( Y \) is both flat and smooth over \( Z \)—for example affine \( n \)-space over \( Z \). Let \( R := \mathcal{O}_{Z,z}, B = \mathcal{O}_{Y,x}, \) and \( A := \mathcal{O}_{X,x} \). Let \( I \) be the kernel of the surjection \( B \rightarrow I \). By Theorem 17, the map \( \overline{d}: I(x) \rightarrow \Omega^1_{Y/Z}(x) \) is injective. Choose a sequence of generators \((b_1, \ldots, b_r)\) for \( I \) such that \((f_1(x), \ldots, f_r(x))\) is a basis for \( I(x) \) Theorem 17 implies that for every \( i \), the subscheme \( X'_i \) of \( Y \) defined by \((b_1, \ldots, b_i)\) is again smooth over \( Z \). We prove that it is flat by induction on \( i \). This is true by assumption if \( i = 0 \), and the general induction step will follow from the case \( i = 1 \). Since \( Y/R \) is smooth, its fibers are regular, and hence \( B \otimes_R k \) is an integral domain. Since the image \( \overline{b} \) of \( b_1 \) in \( B \otimes_R k \) is not zero, \( \overline{b} \) is a nonzero divisor, and since \( B \) is flat, the lemma implies that \( A \) is also flat. This completes the proof.

For the converse, suppose that \( X \rightarrow Z \) is flat and that its geometric fibers are regular. We claim that \( X \rightarrow Z \) is smooth. Again we work locally in a neighborhood of a point \( x \), so we can assume that there is a closed immersion \( i: X \rightarrow Y \) where \( Y \) is smooth over \( Z \)—for example affine space. Suppose \( Y = \text{Spec}(B) \) and \( X = \text{Spec}(A) \), with \( A = B/I \). If \( \overline{z} \) is a geometric point lying over the image \( z \in Z \) of \( X \), then by Corollary 19 we know that \( X \times_z \overline{z} \) is smooth over \( \overline{z} \). Let \( \overline{B} := B \otimes_R k \) and \( \overline{A} := A \otimes_R k \) and let \( \overline{T} \) be the kernel of the map \( B \otimes_R k \rightarrow A \otimes_R k \). Since \( A/R \) is flat, in fact \( \overline{T} = I \otimes_R k_R \). Then

\[
I(x) := I/ot_Bk(x) = I \otimes_B k_R \otimes_{\overline{B}} k(x) \cong \overline{T} \otimes_{\overline{B}} k(x),
\]
and \( \overline{d}: I(x) \to \Omega^1_{B/R}(x) \) identifies with the map \( \overline{I}(x) \to \Omega^1_{B/k_R}(x) \). Tensoring over \( k_R \) with \( k(z) \), we get the corresponding map for the geometric fibers \( X_z \to Y_z \), which is injective. \( \square \)