Homework Assignment #8:

Due April 19

For some of the problems, you will need to use the technique of cohomology and base change. A simplified treatment of the essential facts can be found in the Notes section of the course web page.

1. Let $X/k$ be a smooth proper curve over an algebraically closed field $k$, and let $\mathcal{L}$ be an invertible sheaf on $X$. Suppose that $V := \Gamma(X, \mathcal{L})$ has dimension at least two and generates $\mathcal{L}$. Show that $V$ contains a two-dimensional subspace which generates $\mathcal{L}$.

2. Let $X/k$ be a smooth proper curve over an algebraically closed field $k$, of genus $g$. Prove that there exists a morphism $X \to \mathbb{P}^1$ of degree less than or equal to $g + 1$.

3. Let $X/S$ be a proper and flat morphism, where $S = \text{Spec} \ A$ and $A$ is a noetherian ring. Suppose that all the geometric fibers of $X/A$ are nonempty, connected and reduced. Then the natural map $A \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism.

4. Let $f: X \to S$ be a smooth proper morphism such that for every $s \in S$, the fiber $X(s)$ is a geometrically connected curve of genus one. If $\mathcal{L}$ is a an invertible sheaf on such an $X$, we let $\deg(\mathcal{L})$ be the function which takes a point $s$ to the degree of the restriction of $\mathcal{L}$ to the fiber $X(s)$. This function is always locally constant, we will typically just assume it is constant. Assume for simplicity here that all schemes are noetherian.

(a) Prove that if $\deg(\mathcal{L}) > 0$, then $R^1 f_* \mathcal{L} = 0$, that $f_* \mathcal{L}$ is a locally free sheaf of rank $\deg(\mathcal{L})$, and that for any $g: S' \to S$, the map $g^* f_* \mathcal{L} \to f'_* \mathcal{L}'$ is an isomorphism, where $\mathcal{L}'$ is the inverse image of $\mathcal{L}$ on $X' := X \times_S S'$. It suffice to do the case in which $S$ is affine.
(b) Let $D_1(S) := Div_1(X/S)$ denote the set of isomorphism classes of pairs $(L, s)$, where $L$ is an invertible sheaf on $X$ of degree one and $s$ is an isomorphism $\mathcal{O}_S \to f_* L$. Prove that if $g : S' \to S$ is any map, then $(L', g^* s) \in Div_1(X'/S')$, so that $D_1$ becomes a functor on the category of schemes over $S$.

(c) If $T$ is any scheme, let $Pic(T)$ denote the set of isomorphism classes of invertible sheaves on $T$. If $(X/S, \sigma)$ is an elliptic curve as above, let $Pic(X/S)$ denote the cokernel of the map $Pic(S) \to Pic(X)$. Note that deg factors through $Pic(X/S)$, and let $Pic_d(X/S)$ denote the set of elements of degree $d$. This set is a torsor under the group $Pic_0(X/S)$. If $S' \to S$ is a morphism, base change defines a natural map $Pic(X/S) \to Pic(X'/S')$, compatible with deg. Let $P_1(S)$ denote $Pic_1(X/S)$, a functor in $S$. Define a map $D_1 \to P_1$ by sending the isomorphism class of a pair $(L, s)$ to the image of $L$ in $P_1(S)$. Show that this map is a bijection. (Hint: If $\ell$ is an element of $P_1(S)$, choose some invertible sheaf $L$ of degree one whose isomorphism class in $Pic(X)$ lies in $\ell$. Let $M := f_* L$ and let $L' := Hom(f^* M, L)$. Prove that the natural map $f^* M \to L$ defines a basis $s$ for $f_* L'$, and the class of $(L', s)$ in $P_1(S)$ is $\ell$.)

(d) Let $X/S$ be the functor taking an $S$-scheme $S'$ to the set of $S$-morphisms $S' \to X$, or equivalently, the set of sections $\tau$ of $X'/S'$. If $\tau \in X/S(S)$, the image of $\tau$ is a closed subscheme of $X$, and its ideal sheaf $I_\tau$ is invertible. Prove that the inclusion $I_\tau \to \mathcal{O}_X$ defines a basis for $f_* L_\tau$, where $L_\tau := Hom(I_\tau, \mathcal{O}_X)$. This defines a morphism $X/S \to D_1$. Prove that this morphism an isomorphism. (Hint: If $(L, s) \in D_1(S)$, prove that the image of the map $\mathcal{O}_X \to L$ defined by $s$ is $I\mathcal{L}$, where $I$ is an invertible sheaf of ideals defining a section $\tau$ of $X/S$. )

(e) Now suppose that $\sigma$ is fixed section of $X/S$, which you can use to identify the functor $P_1$ with the functor $P_0$ and hence the functor $X/S$ with the functor $P_0$. Deduce that $X/S$ has a group scheme structure with $\sigma$ as the identity section.