Homework Assignment #5:

Due March 3

1. Let G be an abelian group and let A, B, and C be G-sets. A map $\beta\colon A \times B \to C$, is G-*bilinear* if for all $(a,b) \in A \times B$ and $g \in G$, $(ga,b) = (a,gb) = g(a,b)$. If $\gamma\colon C \to C'$ is a morphism of G-sets and β is bilinear, then so is $\gamma \circ \beta$. Show that there is a universal bilinear map $A \times B \to A \otimes_G B$. Show that the same is true for sheaves of abelian groups on a topological space and sheaves of G-sets. Show that if A and B are G-torsors, then $A \otimes_G B$ is a G-torsor.

2. Let $0 \to G' \xrightarrow{\iota} G \xrightarrow{\pi} G'' \longrightarrow 0$ be an exact sequence of abelian groups on a topological space X. Show that if $c \in \Gamma(X, G'')$, then the presheaf
$$T_c : U \mapsto \{g \in G : \pi(g) = c_{|_U}\}$$
is a sheaf and in fact is naturally a torsor under a suitable action of G'. Show that if $a, b \in G''(X)$, then there is a natural isomorphism of G'-torsors:
$$T_a \otimes_{G'} T_b \xrightarrow{\cong} T_{a+b}.$$

3. With the notation of the previous problem, show that the sequence $H^0(X, G'') \to H^1(X, G') \to H^1(X, G)$ is exact. Hint: If T' is a G'-torsor, then its class in $H^1(X, G)$ is the class of $G \otimes_{G'} T'$, and if this class is trivial, there is a global section t of $G \otimes_{G'} T'$. Let U be an open subset of X and t' a section of T' on U. Then there is a unique $g \in G(U)$ such that $gt' = t_{|_U}$. The class of g in G'' does not depend on t'.

4. Let X denote the real line, viewed as a topological space, let x be a point of X, and let F be the skyscraper sheaf $\mathbf{Z}$ concentrated at x. Prove that there is no epimorphism from a projective object in Ab_X to F.