Homework Assignment #3:

Due February 22

1. Let $R$ be a ring and let $E$ be an $R$-module. Recall that the functor $\mathbf{V}E$ taking an $R$-algebra $A$ to the set $\text{Hom}_R(E, A)$ is representable by the $R$-algebra $S'E$ together with the universal element $v: E \to S'E$ (the inclusion of $E$ into the degree one component of $S'E$). This functor has its values in the category of sets, but in fact it factors naturally through the category of abelian groups. Identifying the scheme $\text{Spec} S'E$ with the functor $\mathbf{V}E$, and using Yoneda, we find a natural group scheme structure $\mathbf{V}E \times_R \mathbf{V}E \to \mathbf{V}E$. Recall that there is a unique derivation $d: S'E \to S'E \otimes E$ such that $d \circ v(e) = 1 \otimes e$ for all $e \in E$ and that this derivation is universal, so that $\Omega^1_{S'E/R} = S'E \otimes E$. If $I$ is a square zero ideal in $A$, how that the action of $\mathbf{V}E(A)$ on itself defined by the group structure is compatible with the the action of $\text{Der}_{S'(E)/R}(I)$. (The only difficulty here is figuring out how to say this, and perhaps the signs.)

2. Let $R$ be a ring and let $E$ be an $R$-module. Recall that $\mathbf{P}E$ is the functor taking to the set of hyperplanes in $A \otimes_R E$, equivalently, the set of isomorphism classes of invertible quotients $\ell: A \otimes_R E \to L$ of $A \otimes_R E$. Recall also that $\mathbf{P}E$ is covered by affine open subfunctors $D^+(e)$, where $D^+(e)$ is the set of isomorphism classes of invertible quotients $\ell$ such that $\ell(1 \otimes e)$ generates $L$; equivalently, the set of $R$-linear maps $v: E \to A$ such that $v(e) = 1$. Let $E'$ be the quotient of $E$ by the submodule of $E$ generated by $e$. We have a closed immersion $\mathbf{P}E' \to \mathbf{P}E$. Show that $D^+(e)$ is the complement of this closed immersion. Show that the group valued functor $\mathbf{V}E'$ acts on the functor $D^+(e)$ and that this action makes $D^+(e)$ is a pseudo-torsor. (This generalizes the fact that $\mathbf{P}^n \setminus \mathbf{P}^{n-1} \cong \mathbb{A}^{n-1}$.)
3. Taking account the degrees, the universal derivation in Problem 1 defines a map
\[ d: S^*E \rightarrow S^*E(-1) \otimes E \]
and multiplication defines a map \( S^*E(-1) \otimes E \rightarrow S^*E \). Show that the composition of these two maps is just multiplication by \( m \) in degree \( m \). Show that the maps and formula remain valid after localization by any homogeneous element \( g \) of \( S^*E \). Deduce that on \( \mathbb{P}E \), there are maps
\[ d_n: \mathcal{O}_{\mathbb{P}E}(n) \rightarrow \mathcal{O}_{\mathbb{P}E}(n-1) \otimes E \]
and that the map \( d_0 \) factors through a map \( d: \mathcal{O}_{\mathbb{P}E} \rightarrow \mathcal{H}(-1) \), where \( \mathcal{H} \subseteq \mathcal{O}_{\mathbb{P}E} \otimes E \) is the universal hyperplane.

4. Show that the map \( d: \mathcal{O}_{\mathbb{P}E} \rightarrow \mathcal{H}(-1) \) constructed above is the universal derivation and defines a canonical isomorphism: \( \Omega^1_{\mathbb{P}E/R} \rightarrow \mathcal{H}(-1) \).

5. Let \( k \) be a field of characteristic not equal to 3, let \( R := k[t] \), and let \( f := X^3 + Y^3 + Z^3 - 3tXYZ \in R[X, Y, Z] \). The ideal of \( R[X, Y, Z] \) generated by \( f \) definex a closed subscheme \( X \) of \( \mathbb{P}^3_R \). At which points of \( X \) does the morphism \( X \rightarrow \text{Spec } R \) fail to be smooth? Answer the analogous question for \( g := t(X^3 + Y^3 + Z^3) - 3XYZ \in R[X, Y, Z] \).