Homework Assignment #1:

Due February 1

1. Let k be a field. For every k-algebra A, let $F(A)$ denote the set of pairs $(a_1, a_2) \in A \times A$ such that either a_1 or a_2 is a unit. Show that F is a subfunctor of $\mathbf{A} \times \mathbf{A}$ but that it is not a Zariski sheaf.

2. Let R be a ring and let E and E' be R-modules. A homomorphism $\theta\colon E' \to E$ induces a morphism of schemes $\mathbf{V}E \to \mathbf{V}E'$. This morphism is a closed immersion if $E' \to E$ is surjective. Let k be a field, let R be the polynomial ring $k[x, y]$, so that the functor h^R on the category of k-algebras can be identified with $\mathbf{A}^2$. Let E be the maximal ideal (x, y), viewed as an R-module. Let E' be the free R-module on two generators e_1, e_2, and let $E' \to E$ be the map sending e_1 to x and e_2 to y. Find equations for $\mathbf{V}E$ inside $\mathbf{V}E' = \mathbf{A}^2 \times \mathbf{A}^2$.

3. Try your hand at Hartshorne's problems 7.1, 7.2, 7.3 (chapter II). **: Hint** for problem 7.3. A map $f\colon X \to \mathbf{P}E$ to projective space is given by an an invertible quotient $\mathcal{O}_X \otimes E \to \mathcal{L}$, and if Y is a subcheme of X, the restiction of f to Y is given by the restriction of this invertible quotient to Y. If this restricition is constant, then the restriction of $\mathcal{L}$ to Y is isomorphic to $\mathcal{O}_Y$. Show that if $\mathcal{O}_Y$ is very ample on Y and Y is projective over a field k, then Y is finite.

4. Think about, and if you like come to office hours to discuss, the following elaboration of problem 2. Begin by thinking about the case in which R is an algebraically closed field.

   Let R be a ring and E an R-module, and recall the functors $\mathbf{V}E$ and $\mathbf{P}E$. We think of $\mathbf{P}E$ as the set of lines (through the origin) in $\mathbf{V}E$. Find a good way to say that an element v of $\mathbf{V}E(B)$ belongs to an element ℓ of $\mathbf{P}E(B)$, and show that that functor $\mathbf{In}E$ which takes B to the set of pairs (v, ℓ) with $v \in \ell$ is representable by a closed subscheme of $\mathbf{V}E \times \mathbf{P}E$. Next, view $\mathbf{In}E$ as a scheme over $\mathbf{P}E$ via the obvious map. Show that its corresponding functor is naturally isomorphic to the functor $\mathbf{V}\mathcal{O}_{\mathbf{P}E(1)}$. Next view $\mathbf{In}E$ as a scheme over $\mathbf{V}E$ via the obvious map. Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{O}_{\mathbf{V}E}$ definining the zero section. Show that its corresponding functor is represented by $Proj\mathcal{G}$, where $\mathcal{G} := \oplus_n \mathcal{I}^n$ (a sheaf of graded $\mathcal{O}_{\mathbf{V}E}$ modules.)

Hints: The functor can be defined as the set of isomorphism classes of pairs (v, H) such that H is a hyperplane in $E \otimes B$ and $v\colon E \to B$ annihilates H, or as the set of isomorphism classes of pairs of maps $\ell : E \otimes B \to L$ and $L \to B$, with ℓ an invertible quotient. A pair $(v, \ell) \in \mathbf{V}E \times \mathbf{P}E$ defines an ideal I of B, namely $I := v(Ker(\ell))$; this ideal defines the subfunctor $\mathbf{In}E$.