Enough injectives

February 17, 2016

Recall that an object $I$ of an abelian category $\mathcal{A}$ is said to be “injective” if the functor $h_I$ from $\mathcal{A}$ to the category of abelian groups is exact. An abelian category $\mathcal{A}$ is said to “have enough injectives” if for every object $M$, there is a monomorphism $M \rightarrow I$ where $I$ is injective. Thus $\mathcal{A}$ has enough injectives iff its dual has enough projectives.

We omit the proof of the following lemma.

**Lemma 1** Let $I$ be an abelian group with the property that for every nonzero integer $m$, multiplication by $m$ on $I$ is surjective. Then $I$ is an injective object in the category of abelian groups.

For example, the group $I := \mathbb{Q}/\mathbb{Z}$ is injective.

**Lemma 2** For every nonzero abelian group $M$, $h_I(M) \neq 0$.

**Proof:** If $M \neq 0$, it contains a nonzero cyclic subgroup $M'$. Since $h_I$ is exact, the map $h_I(M) \rightarrow h_I(M')$ (restriction) is surjective. Thus it suffices to show that $h_I(M') \neq 0$. If $M' \cong \mathbb{Z}$, $h_I(M') \cong \text{Hom}(\mathbb{Z}, I) = \mathbb{Q}/\mathbb{Z} \neq 0$. If $M' \cong \mathbb{Z}/m\mathbb{Z}$, with $m > 1$, $h_I(M') \cong \{ x \in \mathbb{Z}/\mathbb{Q} : mx = 0 \}$, and the class of $1/m$ is nonzero such element. $$

**Theorem 3** For any ring $R$, the category of $R$-modules has enough injectives.

**Proof:** We will first establish the following preliminary result.

**Proposition 4** Let $R$ be a commutative ring. Suppose there is a be an injective $R$-module $J$ with the property that $h_J(M) \neq 0$ whenever $M \neq 0$. Then the category of $R$-modules has enough injectives.

We proceed with a few lemmas.
Lemma 5 Suppose that $J$ is an injective $R$-module such that $h_J(M) \neq 0$ for every nonzero $M$. Then the canonical homomorphism $\eta_M: M \to h_Jh_J(M)$ is injective, for every $M$.

Proof: Let $K$ be the kernel of $\eta_M$. We have a commutative diagram:

\[
\begin{array}{ccc}
K & \overset{i}{\rightarrow} & M \\
\downarrow{\eta_K} & & \downarrow{\eta_M} \\
\overset{h_J}{\rightarrow} & \overset{h_Jh_J(M)}{\rightarrow} & \\
\end{array}
\]

The functor $h_J$ is exact and hence so is $h_Jh_J$. Thus the bottom arrow is injective. Since $\eta_M \circ i = 0$, it follows that $\eta_K = 0$. This implies that $h_J(K) = 0$. Otherwise, there is some nonzero $\phi \in h_J(K)$, and since $\phi \neq 0$, there is a $k \in K$ such that $\phi(k) \neq 0$. But then $\eta_K(k)(\phi) = \phi(k) \neq 0$, a contradiction. Since $h_J(K) = 0$, in fact $K = 0$, by our hypothesis on $I$.

Lemma 6 Any product of injective objects is injective.

Proof: Suppose that $I_s$ is injective for every $s \in S$ and $I := \prod I_s$. By definition, this means that $h_I = \prod_s h_{I_s}$. Since in the category of sets (and hence in the category of abelian groups) a product of surjections is surjective, $h_I$ is exact.

Now we can prove the proposition as follows. For any $R$-module $M$, find a surjective map $F \to h_J(M)$, with $F$ free, say $F = R^{(S)}$. Then we find an injective map $h_Jh_J(M) \to h_J(F) \cong \text{Hom}(R^{(S)}, J) \cong J^S$. By the previous lemma, $J^S$ is injective, and by the one before that, we have an injective map $M \to h_Jh_J(M)$. Then $M \to h_Jh_J(M) \to J^S$ is the desired injection.

To finish the proof of the theorem, it suffices to prove that there is an injective $R$-module $J$ such that $h_J(M) \neq 0$ for every nonzero $M$.

Lemma 7 Let $J := \text{Hom}_Z(R, Q/Z)$. Then $J$ is an injective $R$-module, and for every nonzero $M$, $h_J(M) \neq 0$.

Proof: For any $M$,

\[
\text{Hom}_R(M, J) = \text{Hom}_R(M, \text{Hom}_Z(R, Q/Z)) \cong \text{Hom}_Z(M, Q/Z).
\]

Since $Q/Z$ is injective, it follows that $h_J$ is exact. Furthermore, if $M \neq 0$, it follows that $h_J(M) \neq 0$. 

\[\Box\]