Enough injectives

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Recall that an object I of an abelian category $\mathcal{A}$ is said to be "injective" if the functor h_I from $\mathcal{A}$ to the category of abelian groups is exact. An abelian category $\mathcal{A}$ is said to "have enough injectives" if for every object M, there is a monomorphism $M \to I$ where I is injective. Thus $\mathcal{A}$ has enough injectives iff its dual has enough projectives.

We omit the proof of the following lemma.

Lemma 1 *Let I be an abelian group with the property that for every nonzero integer m, multiplication by m on I is surjective. Then I is an injective object in the category of abelian groups.*

For example, the group $I := \mathbf{Q}/\mathbf{Z}$ is injective.

Lemma 2 *For every nonzero abelian group M, $h_I(M) \neq 0$.*

Proof: If $M \neq 0$, it contains a nonzero cyclic subgroup M'. Since h_I is exact, the map $h_I(M) \to h_I(M')$ (restriction) is surjective. Thus it suffices to show that $h_I(M') \neq 0$. If $M' \cong \mathbf{Z}$, $h_I(M') \cong \mathrm{Hom}(\mathbf{Z}, I) = \mathbf{Q}/\mathbf{Z} \neq 0$. If $M' \cong \mathbf{Z}/m\mathbf{Z}$, with $m > 1$, $h_I(M') \cong \{x \in \mathbf{Z}/\mathbf{Q} : mx = 0\}$, and the class of $1/m$ is nonzero such element. $\square$

Theorem 3 *For any ring R, the category of R-modules has enough injectives.*

Proof: We will first establish the following preliminary result.

Proposition 4 *Let R be a commutative ring. Suppose there is a be an injective R-module J with the property that $h_J(M) \neq 0$ whenever $M \neq 0$. Then the category of R-modules has enough injectives.*

We proceed with a few lemmas.

Lemma 5 *Suppose that J is an injective R-module such that $h_J(M) \neq 0$ for every nonzero M. Then the canonical homomorphism $\eta_M \colon M \to h_J h_J(M)$ is injective, for every M.*

Proof: Let K be the kernel of η_M. We have a commutative diagram:

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "\eta_K"'] & M \arrow[d, "\eta_M"] \\
h_J h_J(K) \arrow[r] & h_J h_J(M)
\end{tikzcd}$$

The functor h_J is exact and hence so is $h_J h_J$. Thus the bottom arrow is injective. Since $\eta_M \circ i = 0$, it follows that $\eta_K = 0$. This implies that $h_J(K) = 0$. Otherwise, there is some nonzero $\phi \in h_J(K)$, and since $\phi \neq 0$, there is a $k \in K$ such that $\phi(k) \neq 0$. But then $\eta_K(k)(\phi) = \phi(k) \neq 0$, a contradiction. Since $h_J(K) = 0$, in fact $K = 0$, by our hypothesis on I.

Lemma 6 *Any product of injective objects is injective.*

Proof: Suppose that I_s is injective for every $s \in S$ and $I := \prod I_s$. By definition, this means that $h_I = \prod_s h_{I_s}$. Since in the category of sets (and hence in the category of abelian groups) a product of surjections is surjective, h_I is exact. □

Now we can prove the proposition as follows. For any R-module M, find a surjective map $F \to h_J(M)$, with F free, say $F = R^{(S)}$. Then we find an injective map $h_J h_J(M) \to h_J(F) \cong \mathrm{Hom}(R^{(S)}, J) \cong J^S$. By the previous lemma, J^S is injective, and by the one before that, we have an injective map $M \to h_J h_J(M)$. Then $M \to h_J h_J(M) \to J^S$ is the desired injection. □

To finish the proof of the theorem, it suffices to prove that there is an injective R-module J such that $h_J(M) \neq 0$ for every nonzero M.

Lemma 7 *Let $J := \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})$. Then J is an injective R-module, and for every nonzero M, $h_J(M) \neq 0$.*

Proof: For any M,

$$\mathrm{Hom}_R(M, J) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})) \cong \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}).$$

Since $\mathbf{Q}/\mathbf{Z}$ is injective, it follows that h_J is exact. Furthermore, if $M \neq 0$, it follows that $h_J(M) \neq 0$. □

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