Flatness

March 6, 2016

Let $\theta: A \to B$ be a homomorphism of rings and let N be a B-module. Then the homomorphism θ allows us to view N as an A-module. (Sometimes this A-module is denote by $\theta_*(N)$; we shall not use this notation here.)

Theorem 1 (Local Criterion of Flatness) *Let $\theta: A \to B$ be a local homomorphism of noetherian local rings. Let k be the residue field of A and let N be a noetherian B-module. Then N is flat as an A-module if and only if $\mathrm{Tor}_1^A(k, N) = 0$.*

Proof: Fix N, and for each A-module M, let $T_N(M) := \mathrm{Tor}_1^A(M, N)$. Recall that if $0 \to K \to F \to M \to 0$ is an exact sequence with F free, then $T_N(M)$ is isomorphic to the kernel of the map $K \otimes_A N \to F \otimes_A N$. From this we can deduce the following facts.

1. $T_N(M)$ has a natural structure of a B-module.
2. If M is finitely generated as an A-module and N is finitely generated as a B-module, then $T_N(M)$ is finitely generated as a B-module. (Note: this uses the hypothesis that A and B are noetherian.)
3. If $0 \to M' \to M \to M'' \to 0$ is exact, then

$$T_N(M') \to T_N(M) \to T_N(M'')$$

   is also exact.
4. T_N commutes with direct limits.

Our hypothesis is that $T_N(k) = 0$, and we want to conclude that $T_N(M) = 0$ for every A-module M. By (4), it suffice to consider finitely generated modules M. Since A is noetherian, any such module is noetherian. Consider the family $\mathcal{F}$ of submodules M' of M such that $T_N(M/M') \neq 0$. Our claim is that the 0 submodule does not belong to $\mathcal{F}$, and so of course it will suffice to prove that $\mathcal{F}$ is empty. Assuming otherwise, we see from the fact that M is noetherian that $\mathcal{F}$ has a maximal element M_0. Let $\overline{M} := M/M_0$. Then $T_N(\overline{M}) \neq 0$, but $T_N(\overline{M}) = 0$ for every nontrivial quotient $\overline{M}''$ of $\overline{M}$. We shall see that this leads to a contradiction. To simply the notation, we replace M by $\overline{M}$. In other words, it suffices to prove that $T_N(M) = 0$ under the additional assumption that $T_N(M'') = 0$ for every proper quotient of M. Then for every nontrivial submodule M' of M, we have an exact sequence:

$$T_N(M') \to T_N(M) \to 0 \tag{1}$$

We argue case by case as follows:
Case 1: $Ann(M) = \mathrm{m}_A$. Choose some nonzero $x \in M$ and let M' be the submodule it generates. Then $M' \cong A/Ann(x) = A/\mathrm{m}_A$, so by hypothesis $T_N(M') = 0$. By the sequence (1) above, it follows that $T_N(M) = 0$.

Case 2: There exists some $a \in \mathrm{m}_A \setminus Ann(M)$. Let M' be the kernel of multiplication by a on M. Then we have exact sequences:

$$\begin{array}{rcl} 0 \to M' \to & M & \to aM \to 0 \\ 0 \to aM \to & M & \to M/aM \to 0, \end{array}$$

and hence also exact sequences:

$$\begin{array}{rcl} T_N(M') \to & T_N(M) & \to T_N(aM) \\ T_N(aM) \to & T_N(M) & \to T_N(M/aM). \end{array}$$

Since $aM \neq 0$, $T_N(M/aM)$ is a proper quotient of M, and hence $T_N(M/aM) = 0$, by hypothesis.

Case 2a: If $M' \neq 0$, then the first sequence above shows that aM is also a proper quotient of M, and hence also $T_N(aM) = 0$, and then the last sequence implies that $T_N(M) = 0$.

Case 2b: If $M' = 0$, multiplication by a on M is injective, and we find exact sequences

$$\begin{array}{ccccc} 0 \to M & \xrightarrow{a} & M & \longrightarrow & M/aM \to 0 \\ T_N(M) & \xrightarrow{a} & T_N(M) & \longrightarrow & 0 \end{array}$$

Thus multiplication by a on $T_N(M)$ is surjective. But $T_N(M)$ is a finitely generated B-module, and multiplication by a on this module is the same as multiplication by $\theta(a)$. Since $\theta(a)$ belongs to the maximal ideal of B, Nakayama's lemma implies that $T_N(M) = 0$, as required. $\square$

Proposition 2 *Let A be a ring, let I be an ideal of A, and let M be an A-module. Suppose that M/IM is flat as an A/I-module and also that $\mathrm{Tor}_1^A(A/I, M) = 0$. Then $\mathrm{Tor}_1^A(A/J, M) = 0$ for every ideal J containing I.*

Proof: Let $0 \to K \to F \to M \to 0$ be an exact sequence, with F free. Since $\mathrm{Tor}_1^A(A/I, M) = 0$, the sequence

$$0 \to K \otimes_A (A/I) \to F \otimes A/I \to M/IM \to 0$$

is still exact. This is an exact sequence of A/I-modules, and since M/IM is flat as an A/I-module, the sequence remains exact if we tensor over A/I with A/J:

$$0 \to (K \otimes_A A/I)) \otimes_{A/I} A/J \to (F \otimes A/I) \otimes_{A/I} A/J \to M/IM \otimes_{A/I} A/J \to 0$$

is still exact. But for any A-module, $(M \otimes_A A/I) \otimes_{A/I} A/J \cong M \otimes_A A/J$, so this last sequence can be identified with:

$$0 \to K \otimes_A A/J \to F \otimes A/J \to M \otimes A/J \to 0.$$

The injectivity on the left implies that $\mathrm{Tor}_1^A(A/J, M) = 0$. □

Theorem 3 (Criterion of Flatness along the Fiber) *Let $R \to A$ and $A \to B$ be local homomorphisms of noetherian local rings, and let k be the residue field of R, and let N be a finitely generated B-module. If N is flat over R and $N \otimes_R k$ is flat over $A \otimes_R k$, then N is flat over A. If in addition $N \neq 0$, then in fact A is flat over R.*

Proof: Let m_R be the maximal ideal of R and let I be the ideal of A generated by its image in A. Then we have a surjective map $\mathrm{m}_R \otimes_R A \to I$, and hence also a surjective map:

$$\mathrm{m}_R \otimes_R A \otimes_A N \to I \otimes_A N.$$

Since $\mathrm{m}_R \otimes_R A \otimes_A N \cong \mathrm{m}_R \otimes_R N$, we find maps

$$\mathrm{m}_R \otimes_R N \xrightarrow{f} I \otimes_A N \xrightarrow{g} N$$

We have just seen that f is surjective. On the other hand, the kernel of $g \circ f$ is $\mathrm{Tor}_R^1(k, N) = 0$, so $g \circ f$ is injective. Then it follows that f is also injective, hence an isomorphism, and hence that g is injective. The kernel of g is $\mathrm{Tor}_1^A(A/I, N)$, and we can conclude that this Tor vanishes. Furthermore, $N/IN \cong N \otimes_A (A/I) \cong N \otimes_A A \otimes_R k \cong N \otimes_R k$, and since N is flat over

R, $N \otimes_R k$ is flat over $A \otimes_R k \cong A/I$. By the previous result, it follows that $\operatorname{Tor}_1^A(A/J, N) = 0$ for every ideal J containing I and in particular for J equal to the maximal ideal of A. By the local criterion for flatness, this implies that N is flat over A.

If $N \neq 0$, then since it is flat over A and $A \to B$ is a local homomorphism, in fact N is faithfully flat as an A-module. Since N is by assumption flat over R, it follows that A is also flat over R. We omit the proof. □