Depth and CohomoIogy

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Let A be a ring and E a nonzero A-module. A sequence of elements $(a_1, \ldots, a_r)$ is *E-regular* if $E/(a_1, \ldots, a_r)E \neq 0$ and for all i, multiplication by a_i acts injectively $E/(a_1, \ldots, a_{i-1})E$. If I is an ideal of A, $\operatorname{depth}_I(E)$ is the maximum length of an E-regular sequence of elements in I. If there are no such sequences we set $\operatorname{depth}_I(E) = 0$, and if there is no bound on the length of such a sequence or if $E/IE = 0$ we set $\operatorname{depth}_I(E) = \infty$. If P and Q are prime ideals of A with $P \subseteq Q$, $\dim(Q, P)$ is the maximum length of a chain of prime ideals joining P and Q, and $\operatorname{dist}(Q, P)$ is the minimum length of a saturated chain of prime ideals joining P and Q. If A is a local ring and $\mathfrak{m}$ is its maximal ideal, then $\operatorname{depth}_{\mathfrak{m}}(E)$ is often just written as $\operatorname{depth}(E)$.

Lemma 0.1 *With above conventions,*

1. *If $I \subseteq \sqrt{J}$, then $\operatorname{depth}_I(E) \leq \operatorname{depth}_J(E)$.*
2. *If Q is a prime ideal, then $\operatorname{depth}_Q(E) \leq \operatorname{depth}(E_Q)$, where in the latter E_Q is regarded as a module over the local ring A_Q.*

$\square$

Theorem 0.2 *Assume that A is a noetherian ring, that E is a noetherian A-module, and that I is an ideal of A.*

1. *If Q is a prime ideal belonging to the support of E/IE, then*

$$\operatorname{depth}_I(E) \leq \dim_Q E.$$

2. $\operatorname{depth}_I(E) = \inf\{i : \operatorname{Ext}^i_A(A/I, E) \neq 0\}$.

3. If P is an associated prime of E and Q contains P, then

$$\operatorname{depth}_Q(E) \leq \operatorname{dist}(Q, P).$$

Proof: We give here only the proof of (3). Note first that $E/QE \neq 0$, because Q contains an associated prime of E and E is finitely generated. Since $\operatorname{dist}(Q, P)$ does not changen if we replace Q by A_Q and Q and P by their corresponding primes in A_Q, (2) of Lemma 0.1 implies that we may assume without of generality that A is local and that Q is its maximal ideal.

We argue by induction on $d := \operatorname{dist}(Q, P)$. If $d = 0$, then $Q = P$ and hence is associated to E. In this case no elemenet of Q acts injectively on E, and hence $\operatorname{depth}(E) = 0$. Suppose that $d > 0$ and that $Q = Q_0 \supset \cdots Q_d = P$ is a saturated chain of distinct primes joining f P and Q, with d minimal. Then $Q_1 \supset \cdots P$ is a saturated chain of prime ideals joining Q_1 and P, and there can be no shorter such chain. Thus $\operatorname{dist}(Q_1, P) = d - 1$, and by the induction assumption, we have

$$\operatorname{depth}_{Q_1}(E) \leq d - 1 \tag{1}$$

Choose $x \in Q \setminus Q_1$ and let $J := (Q_1, x)$. Thus $Q_1 \subsetneq J \subseteq Q$. Since there are no prime ideals between Q_1 and Q, Q is the only prime ideal containing J, and hence Q is nilpotent modulo J. It follows that $\sqrt{J} = Q$ and hence by (1) of Lemma 0.1,

$$\operatorname{depth}_J(E) = \operatorname{depth}_Q(E). \tag{2}$$

By Lemma (2) below, $\operatorname{depth}_{Q_1}(E) \geq \operatorname{depth}_J(E) - 1$. Combining this inequality with (1) and (2), we find

$$\operatorname{depth}_Q(E) = \operatorname{depth}_J(E) \leq \operatorname{depth}_{Q_1}(E) + 1 \leq (d-1) + 1 = d,$$

proving the theorem. It remains only to prove the lemma below.

Lemma 0.3 *With the notation above,*

$$\operatorname{depth}_{Q_1}(E) \geq \operatorname{depth}_J(E) - 1$$

The lemma asserts that $\operatorname{Ext}^i(A/Q_1, E) = 0$ for $i < \operatorname{depth}_J(E) - 1$. To prove this, note that since Q_1 is a prime ideal and $x \in Q \setminus Q_1$, multiplication by x on A/Q_1 is injective. Since $J = Q_1 + (x)$, we find a short exact sequence

$$0 \longrightarrow A/Q_1 \xrightarrow{x} A/Q_1 \longrightarrow A/J \longrightarrow 0,$$

and consequently a long exact sequence:

$$\mathrm{Ext}^i(A/J, E) \longrightarrow \mathrm{Ext}^i(A/Q_1, E) \xrightarrow{x} \mathrm{Ext}^i(A/Q_1, E) \longrightarrow \mathrm{Ext}^{i+1}(A/J, E).$$

If $i < \mathrm{depth}_J(E) - 1$, then $i + 1 < \mathrm{depth}_J(E)$, and hence $\mathrm{Ext}^{i+1}(A/J, E) = 0$. Then it follows from the exact sequence above that multiplication by x on $\mathrm{Ext}^i(A/Q_1, E)$ is surjective. But $\mathrm{Ext}^i(A/Q_1, E)$ is finitely generated over the local ring A, and x belongs to the maximal ideal of A, and it follows by Nakayama's lemma that $\mathrm{Ext}^i(A/Q_1, E) = 0$, as required. □

A ring is said to be *catenary* if for any pair of prime ideals with $P \subseteq Q$, $\mathrm{dist}(Q, P) = \dim(Q, P)$. Since every prime ideal contains a minimal prime and is contained in a maximal prime, it is enough to verify this condition whenever Q is maximal and P is minimal. The quotient of a catenary ring is necessarily catenary.

A noetherian local ring R is *Cohen-Macaulay* if $\mathrm{depth}(R) = \dim(R)$. Statement (3) of the previous theorem implies that every associated prime of such a ring R is minimal, and since every minimal prime is also associated, it follows that R is catenary. Hence any quotient of a Cohen-Macaulay local ring is catenary.