Let $A$ be a ring and $E$ a nonzero $A$-module. A sequence of elements $(a_1, \ldots, a_r)$ is $E$-regular if $E/(a_1, \ldots, a_r)E \neq 0$ and for all $i$, multiplication by $a_i$ acts injectively $E/(a_1, \ldots, a_{i-1})E$. If $I$ is an ideal of $A$, depth$_I(E)$ is the maximum length of an $E$-regular sequence of elements in $I$. If there are no such sequences we set depth$_I(E) = 0$, and if there is no bound on the length of such a sequence or if $E/IE = 0$ we set depth$_I(E) = \infty$. If $P$ and $Q$ are prime ideals of $A$ with $P \subseteq Q$, dim($Q, P$) is the maximum length of a chain of prime ideals joining $P$ and $Q$, and dist($Q, P$) is the minimum length of a saturated chain of prime ideals joining $P$ and $Q$. If $A$ is a local ring and $m$ is its maximal ideal, then depth$_m(E)$ is often just written as depth($E$).

Lemma 0.1 With above conventions,

1. If $I \subseteq \sqrt{J}$, then depth$_I(E) \leq$ depth$_J(E)$.

2. If $Q$ is a prime ideal, then depth$_Q(E) \leq$ depth($E_Q$), where in the latter $E_Q$ is regarded as a module over the local ring $A_Q$.

\[ \square \]

Theorem 0.2 Assume that $A$ is a noetherian ring, that $E$ is a noetherian $A$-module, and that $I$ is an ideal of $A$.

1. If $Q$ is a prime ideal belonging to the support of $E/IE$, then 
   \[ \text{depth}_I(E) \leq \text{dim}_Q E. \]

2. \[ \text{depth}_I(E) = \inf\{i : \text{Ext}_A^i(A/I, E) \neq 0\}. \]
3. If $P$ is an associated prime of $E$ and $Q$ contains $P$, then

\[ \text{depth}_Q(E) \leq \text{dist}(Q, P). \]

Proof: We give here only the proof of (3). Note first that $E/QE \neq 0$, because $Q$ contains an associated prime of $E$ and $E$ is finitely generated. Since $\text{dist}(Q, P)$ does not change if we replace $Q$ by $A_Q$ and $Q$ and $P$ by their corresponding primes in $A_Q$, (2) of Lemma 0.1 implies that we may assume without generality that $A$ is local and that $Q$ is its maximal ideal.

We argue by induction on $d := \text{dist}(Q, P)$. If $d = 0$, then $Q = P$ and hence is associated to $E$. In this case no element of $Q$ acts injectively on $E$, and hence depth$(E) = 0$. Suppose that $d > 0$ and that $Q = Q_0 \supset \cdots \supset Q_d = P$ is a saturated chain of distinct primes joining $f$ and $Q$, with $d$ minimal. Then $Q_1 \supset \cdots \supset P$ is a saturated chain of prime ideals joining $Q_1$ and $P$, and there can be no shorter such chain. Thus $\text{dist}(Q_1, P) = d - 1$, and by the induction assumption, we have

\[ \text{depth}_{Q_1}(E) \leq d - 1 \]  

(1)

Choose $x \in Q \setminus Q_1$ and let $J := (Q_1, x)$. Thus $Q_1 \subsetneq J \subseteq Q$. Since there are no prime ideals between $Q_1$ and $Q$, $Q$ is the only prime ideal containing $J$, and hence $Q$ is nilpotent modulo $J$. It follows that $\sqrt{J} = Q$ and hence by (1) of Lemma 0.1,

\[ \text{depth}_J(E) = \text{depth}_Q(E). \]  

(2)

By Lemma (2) below, $\text{depth}_{Q_1}(E) \geq \text{depth}_J(E) - 1$. Combining this inequality with (1) and (2), we find

\[ \text{depth}_Q(E) = \text{depth}_J(E) \leq \text{depth}_{Q_1}(E) + 1 \leq (d - 1) + 1 = d, \]

proving the theorem. It remains only to prove the lemma below.

Lemma 0.3 With the notation above,

\[ \text{depth}_{Q_1}(E) \geq \text{depth}_J(E) - 1 \]

The lemma asserts that $\text{Ext}^i(A/Q_1, E) = 0$ for $i < \text{depth}_J(E) - 1$. To prove this, note that since $Q_1$ is a prime ideal and $x \in Q \setminus Q_1$, multiplication by $x$ on $A/Q_1$ is injective. Since $J = Q_1 + (x)$, we find a short exact sequence

\[ 0 \longrightarrow A/Q_1 \xrightarrow{x} A/Q_1 \longrightarrow A/J \longrightarrow 0, \]
and consequently a long exact sequence:

$$\operatorname{Ext}^i(A/J, E) \longrightarrow \operatorname{Ext}^i(A/Q_1, E) \longrightarrow \operatorname{Ext}^{i+1}(A/J, E).$$

If $i < \text{depth}_J(E) - 1$, then $i + 1 < \text{depth}_J(E)$, and hence $\operatorname{Ext}^{i+1}(A/J, E) = 0$. Then it follows from the exact sequence above that multiplication by $x$ on $\operatorname{Ext}^i(A/Q_1, E)$ is surjective. But $\operatorname{Ext}^i(A/Q_1, E)$ is finitely generated over the local ring $A$, and $x$ belongs to the maximal ideal of $A$, and it follows by Nakayama’s lemma that $\operatorname{Ext}^i(A/Q_1, E) = 0$, as required.

A ring is said to be catenary if for any pair of prime ideals with $P \subseteq Q$, $\text{dist}(Q, P) = \text{dim}(Q, P)$. Since every prime ideal contains a minimal prime and is contained in a maximal prime, it is enough to verify this condition whenever $Q$ is maximal and $P$ is minimal. The quotient of a catenary ring is necessarily catenary.

A noetherian local ring $R$ is Cohen-Macaulay if $\text{depth}(R) = \text{dim}(R)$. Statement (3) of the previous theorem implies that every associated prime of such a ring $R$ is minimal, and since every minimal prime is also associated, it follows that $R$ is catenary. Hence any quotient of a Cohen-Macaulay local ring is catenary.