Colimits and Localization

January 30, 2016

For some reason, calculating colimits, say in the category of sets, seems to be more difficult that calculating limits: forming colimits require taking the “quotient” by an equivalence relation which can be difficult to make explicit. The calculations are much easier when the following conditions are fulfilled.

Definition A category $I$ is said to be filtering if it satisfies the following conditions:

1. It is not empty.
2. For any two objects $i$ and $j$, there exists arrows $a$ and $b$ such that $s(a) = i$, $s(b) = j$, and $t(a) = t(b)$:
3. For any two arrows $a$ and $b$ with the same source and target, there exists an arrow $c$ such that $ca = cb$.

For example, the category $\mathbb{N}$ of natural numbers is filtering. Its opposite, is also filtering but it in a trivial way, in that it has final object. (Observe that, in general, if $I$ has a final object $o$, then for every $i$ there is a unique arrow $a_i: i \to o$, and if $C$ is an $I$-system, $\text{colim}(C)$ is just $C_0$, and $\{C_a : i \in I\}$ is the universal family.)

Colimits over filtering categories are sometimes called direct limits. The following result gives an explicit description of colimits over filtering categories in the category of sets.

Theorem Let $I$ be a filtering category and let $S$ be an $I$-system of sets. Let $S_s$ denote the disjoint union of all the sets $S_i$ and let $E \subseteq S_s \times S_s$ be the set of pairs $(s_i, s_j) \in S_s \times S_s$ such that there exist arrows $a$ and $b$ in $I$ such that $\text{Source}(a) = i$, $\text{Source}(b) = j$, $\text{Target}(a) = \text{Target}(b)$, and $S_a(s_i) = S_b(s_j)$. Then $E$ is an equivalence relation, the quotient $S_s/E$ is a
colimit of \(S\), and the evident family of maps \(q_i: S_i \to S_*/E\) is a universal compatible family.

**Corollary** If \(I\) is filtering, forming the colimit over \(I\) commutes with the forget functor from the category of abelian groups to the category of sets.

**Example.** Let \(M\) be a monoid and let \(S\) be an \(M\)-set. The transporter category of \(S\) is the category whose objects are the elements of \(S\), and for \(s, s' \in S\), the arrows from \(s\) to \(s'\) are the elements \(m\) of \(M\) such that \(ms = s'\).

Then multiplication in \(M\) defines a composition law to make this collection of objects and arrows into a category. Let us check that if \(M\) is commutative and if \(S\) is \(M\), acting on itself, then the transporter category is filtering. It is not empty because \(M\) contains a unit element. Suppose that \(s\) and \(t\) are elements of \(S\). Then \(st = ts\), and \(t\) maps \(s\) to \(ts = st\) and \(s\) maps \(t\) to \(st = ts\), so (2) is satisfied. Suppose next that \(a\) and \(b\) are arrows from \(s\) to \(t\), so that \(as = t\) and \(bs = t\). Let \(u := st\). Then \(s\) is an arrow from \(t\) to \(u\) and \(sa = sb\), so (3) is also satisfied. It is however not true that the transporter category of every \(M\)-set is filtering. A not so obvious theorem asserts that the transporter category of an \(M\)-set \(S\) is filtering if and only if \(S\) is a direct limit (filtered colimit) of free \(M\)-sets.

**Localization** Now let \(C\) be a category, and suppose that all filtered colimits exist in \(C\). Let \(E\) be an object of \(C\) and let \(S\) be a commutative monoid acting by endomorphisms of \(E\). For \(s \in S\) we denote the corresponding endomorphism of \(E\) by \(\mu_E(s)\). For example, \(C\) might be the category of modules over a ring \(R\), \(S\) might be a submonoid of the multiplicative monoid of \(R\) and \(E\) an \(R\)-module. Let \(I\) be the transporter category of \(S\), viewed as an \(S\)-set acting on itself, and define an \(I\)-diagram \(E_i\) in \(C\) by sending every \(i \in I\) to \(E\) and every arrow \(a\) to \(\mu_E(a): E_i = E \to E_{ai} = E\). Let \(\{q_i: E_i \to L\}\) be the colimit, i.e., the universal family of maps satisfying the compatibility condition

\[
q_i = q_{ti} \circ \mu_E(t)
\]

for all \(i \in I\) and all \(t \in S\). The commutativity of \(S\) implies that, for every \(s \in S\), \(\mu_E(s)\) defines a map \(E_i \to E_i\) which is compatible with all the maps \(q_i\). By the universal property of \(L\), we find a unique map \(\mu_L(s): L \to L\) such that \(\mu_L(s) \circ q_i = q_i \circ \mu_E(s)\) for every \(i\).

**Theorem** The object \(L\) above has the following properties.

1. For every \(s \in S\), the arrow \(\mu_L(s): L \to L\) is an isomorphism.
2. The map $q_0: E \to L$ is compatible with the actions of $S$.

3. If $\alpha: E \to F$ is another arrow in $C$, with $S$ acting as isomorphisms on $F$, then there is a unique arrow $\theta: L \to F$ such that $\theta \circ q_0 = \alpha$.

**Proof:** To construct an inverse to $\mu_L(s)$, we use the following tricky argument. For each $i \in I$, recall that $E_i = E = E_{is}$, and so $q_{is}$ can also be viewed as a map $\tilde{q}_i: E_i \to L$. Then $\{\tilde{q}_i: E_i \to L\}$ is another family of compatible maps, which then induces a map $\tilde{s}: L \to L$, uniquely determined by the fact that $\tilde{s} \circ q_i = \tilde{q}_i$ for all $i$. We claim that $\tilde{s} \circ \mu_L(s) = \mu_L(s) \circ \tilde{s} = \text{id}_L$. To check this, it is enough to see that the equalities hold after composing both sides with $q_i$. We compute:

$$\mu_L(s) \circ \tilde{s} \circ q_i = \mu_L(s) \circ \tilde{q}_i$$

$$= \mu_L(s) \circ q_{is}$$

$$= q_{is} \circ \mu_E(s)$$

$$= q_i,$$

using equation (1) and the commutativity of $S$. Similarly;

$$\tilde{s} \circ \mu_L(s) \circ q_i = \tilde{s} \circ q_i \circ \mu_E(s)$$

$$= q_{is} \circ \mu_E(s)$$

$$= q_i,$$

Statement (2) is built in the construction. For (3), suppose that $\alpha$ is given. Construct the $I$-diagram $F^*$ in the same way that we did for $E$. By hypothesis, all the arrows $F_i \to F_{is}$ are isomorphisms. Note that the identity element 1 of $S$ defines an initial object 0 of $I$: for every $i \in I$, there is a unique arrow $a_i: 0 \to i$ (namely $i$). Let $q_i' := F_{a_i^{-1}}: F_i \to F_0$. Then this family is compatible, and it follows that the composition $E_i \to F_i \to F_0 = F$ also forms a compatible family. This family induces a morphism $L \to F$, and we leave the rest of the verifications to reader. 

Let us return to the more down-to-earth case of $R$-modules. We should compare the categorical construction given here with the usual construction of a localization of an $R$-module $E$ by a multiplicative subset $S$ of $R$. Typically this is done by taking the quotient of the product $E \times S$ by the equivalence relation given by:

$$(e, s) \sim (e', s') \iff \text{there exist } s'' \in S : s'' s' e = s'' s e' \quad (2)$$
(We think of the equivalence class of \((e, s)\) as a ratio \(e/s\).) Now the above construction says to take the colimit over the \(I\)-diagram \(E\). Let us apply the construction in the theorem, which says to form the disjoint union of the sets \(E_i\) and then divide by a certain equivalence relation. Since the objects of \(I\) are the same as the elements of \(S\) and since \(E_i = E\) for every \(i\), this disjoint union is exact the same as \(E \times S\). What is the equivalence relation? It says

\[(e, s) \sim (e', s') \iff \text{there exist } t, t' \in S : ts = t's' \text{ and } te = t'e' \quad (3)\]

It is perhaps not obvious that the equivalence relations (2) and (3) are the same. Suppose that (3) holds. Then

\[ts'e = s'te = s't'e' = t's'e' = tse'\]

Thus if we take \(s'' := t\), we see that (2) holds. Suppose that (2) holds. Then take \(t' := s''s\) and \(t := s''s'\). Then

\[ts = s''s's = s''ss' = t's' \text{ and } te = s''s'e = s''se' = t'e'\]