Associated Primes

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Definition 1 *Let R be a ring and let E be an R-module.*

1. *The annihilator of an element x of E is the set $Ann(x)$ of all $a \in R$ such that $ax = 0$. It is an ideal of R, the kernel of the map $R \to E$ sending 1 to x.*

2. *The annihilator of E is the set of all a in R such that $ax = 0$ for all x in E. It is also an ideal of R, and is the intersection of the ideals $Ann(x) : x \in E$.*

3. *The support of E is the set $Supp(E)$ of all prime ideals P such that the localizaiton E_P of E at P is not zero.*

4. *A prime ideal P is associated to E if there is some element x of E such that $Ann(x) = P$. The set of such prime ideals is denoted by $Ass(E)$. Thus a prime ideal P belongs to E if and only if there is an injection $A/P \to E$.*

Proposition 2 *If E is an R-module and P is a prime ideal of R such that $P \in Supp(E)$, then P contains $Ann(E)$. The converse holds if E is finitely generated.*

Proof: If there is some $a \in Ann(E) \setminus P$, then $ax = 0$ for every $x \in E$, hence x maps to zero in the localization E_P, and hence $E_P = 0$. For the converse, suppose that $x_1, \ldots, x_n$ is a finite set of generators for E and that P is a prime ideal containing $Ann(E)$. Since E is generated by $x_1, \ldots, x_n$, $Ann(E)$ is the intersection of the ideals $Ann(x_i)$ for $i = 1, \ldots, i_n$. We claim that P contains $Ann(x_i)$ for some i. Otherwise there is some $a_i \in Ann(x_i) \setminus P$ for every i, and $a := \prod_i a_i \in Ann(E) \setminus P$, a contradiction. But if P contains $Ann(x_i)$, $ax_i \neq 0$ for every $a \in R \setminus P$, and hence the image of x_i in E_P is not zero. Thus $P \in Supp(E)$. $\square$

To see that the hypothesis of finite generation is not superflous in the above proposition, consider the following example. Take $R = \mathbf{Z}$ and $E = \oplus_p \mathbf{Z}/p\mathbf{Z}$. Then $Ann(E)$ is the zero ideal, but the localization of E by the zero ideal is $E \otimes \mathbf{Q} = 0$.

Proposition 3 *Let R be a commutative ring and E an R-module. Suppose that the localization E_P of E at P is zero for every maximal ideal P of R. Then $E = 0$.*

Proof: Let x be an element of E. For each P, let $\lambda_P\colon E \to E_P$ be the localization homomorphism. Then $\lambda_P(x) = 0$. This means that there is some $s_P \in R \setminus P$ such that $s_P x = 0$. Let I be the ideal of R generated by the set of all these elements s_P. For every maximal ideal P of R, the ideal I contains $s_P \notin P$, and hence $I \not\subseteq P$. Since I is not contained in any maximal ideal of R, I is not a proper ideal, hence $1 \in I$. This implies that there exists a finite sequence $s_{P_1}, \cdots, s_{P_m}$ and a elements $a_1, \ldots a_m$ such that $1 = a_1 s_{P_1} + \cdots + a_m s_{P_m}$. Then $x = a_1 s_{P_1} x + \cdots + a_m s_{P_m} x = 0$. □

Proposition 3 implies that the support of E is empty if and only if $E = 0$.

Corollary 4 *Let E be an R-module and a an element of R. Then the following are equivalent.*

1. *The localization E_a of E at a vanishes.*
2. *For every $x \in E$, there exists an n such that $a^n x = 0$. (Then a is said to be locally nilpotent on E.)*
3. *The element a belongs to every P in the support of E.*

Proof: The equivalence of (1) and (2) is clear. To prove that (2) implies (3), suppose P is a maximal ideal in the support of E. Then $E_P \neq 0$, so for some $x \in E$, $\lambda_P(x) \neq 0$, and hence for every $s \notin P$, $sx \neq 0$. Since $a^n x = 0$ for some n, $a^n \in P$, hence $a \in P$. To prove that (3) implies (1), suppose that $E_a \neq 0$. We can view E_a as a module over the ring R_a obtained by localizing the ring R by a. Then by Proposition 3, applied to the R_a-module E_a, there exists a prime ideal in the support of E_a. This prime ideal is the localization P_a of some prime ideal P in R not containing a. Since $(E_a)_{P_a} = E_P$, we see that $E_P \neq 0$; *i.e.*, P belongs to the support of E. By hypothesis, $a \in P$, a contradiction. □

Note that $Ass(E) \subseteq Supp(E)$, because if $A/P \to E$ is injective, the localized map $(A/P)_P \to E_P$ is injective, and $(A/P)_P \neq 0$. (In fact $(A/P)_P$ is a field).

Proposition 5 *If R is noetherian and $E \neq 0$, then $Ass(E) \neq \emptyset$. Moreover, a prime ideal P belongs to the support of E if and only if P contains a prime associated to E. In particular, every minimal element of $Supp(E)$ belongs to $Ass(E)$.*

Proof: For the first part, see Lang. Suppose $Q \in Ass(E)$ and $Q \subseteq P$. Then $T := R \setminus Q$ contians $S := R \setminus P$, so that E_Q is a localization of E_P. Since $E_Q \neq 0$, it follows that $E_P \neq 0$. Conversely, suppose that $E_P \neq 0$. Then E_P has an associated prime Q. Thus there is some element y of E_P such that $Ann(y) = Q$. Evidently $Q \subseteq P$; otherwise there is some $q \in Q$ which acts as an isomorphism on E_P and kills y, contradicting the fact that $y \neq 0$. Say $y = \lambda_P(x)/s$ with $s \in S := R \setminus P$. Since s acts as an isomorphism on E_P, $Ann(sy) = Ann(y)$, so without loss of generality $y = \lambda(x)$. Let $q_1, \cdots, q_m$ be a finite set of generators for Q. Then $\lambda(q_i x) = 0$, hence there is some $s_i \in S$ such that $s_i q_i x = 0$. Let s be the product of all these s_i. Then $q_i s x = 0$ for all i, and hence $qsx = 0$ for all q, so $Q \subseteq Ann(sx)$. On the other hand, if $ax = 0$, then $a\lambda(sx) = 0$, hence $sa\lambda(x) = 0$ and hence $a\lambda(x) = 0$, so $a \in Q$. Thus $Ann(sx) = Q$ and $Q \in Ass(E)$. □

Proposition 6 *Let E be a module over a noetherian ring R and let a be an element of R*

1. *Multiplication by a on E is locally nilpotent iff a belongs to every associated prime of E.*

2. *Multiplication by a on E is injective iff a belongs to no associated prime of E.*

Proof: It follows from Proposition 5 that a belongs to to every associated prime of E iff it belongs to every prime in the support of E. By Corollary 4, this is true iff multiplication by a on E is locally nilpotent. This proves (1).

For (2), suppose first that $a \in P \in Ass(E)$. Then there is a nonzero x in E such that $Ann(x) = P$. Thus $ax = 0$ and $x \neq 0$, so multiplication by a is not injective. Suppose for the converse that $ax = 0$ for some $x \neq 0$. Let E' be the set of multiples of x. Then $Ass(E') \neq 0$, so there is some multiple of x, say bx, such that $Ann(bx)$ is a prime ideal P. Then $P \in Ass(E)$ and $abx = 0$, so $a \in P$. □

Proposition 7 *Suppose that R is noetherian and M is a noetherian R-module. Then M admits a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots M_n = M$ such that $M_i/M_{i-1} \cong A/P_i$ for some prime ideal P_i. Morever, $Ass(M) \subseteq \{P_1, \cdots, P_n\}$. In particular, $Ass(M)$ is finite.*

We omit the proof.

A nonzero module E is said to be *coprimary* if for every $a \in A$, multiplication by a is either locally nilpotent or injective. If R is noetherian, then it follows from Proposition 6 that E is coprimary iff it has a unique associated prime.