Math 250B Midterm:

March 1, 2016

1. (a) Let \( E \) be an \( R \)-module. Prove that the functor \( T_E : M \to M \otimes E \) commutes with all colimits.

   **Solution:** This is because \( T_E \) has a right adjoint, \( h^E \), taking \( N \) to \( \text{Hom}(E,N) \). Specifically, \( \text{Hom}(M \otimes E,N) \cong \text{Hom}(M,\text{Hom}(E,N)) \).

(b) Prove that a direct limit (filtering colimit) of flat modules is flat.

   **Solution:** Suppose that \( E \) is a direct system of flat \( R \)-modules and that \( M' \to M \) is an injection of \( R \)-modules. Let \( E := \lim \to E_i \).

Then each \( E_i \otimes M' \to E_i \otimes M \) is injective, since each \( E_i \) is flat. Since the direct limit of injections is injective and since tensor products commute with direct limits, we find that the map \( E \otimes M' \to E \otimes M \) is injective, and hence that \( E \) is flat.

(c) Show that a more general colimit of flat modules need not be flat. (Give a counterexample.)

   **Solution:** Let \( R \) be the ring of polynomials in one variable \( x \) over a field \( k \). Then \( R \) is a free module over itself, hence flat. The coequalizer of \( 0 \) and multiplication by \( x \) on \( R \) is the quotient \( R/(x) \), which is not flat, because \( (x)/(x^2) \to R/(x) \) is not injective.

2. Let \( R \) be a commutative ring with identity, let \( \mathcal{M}_R \) be the category of \( R \)-modules, and let \( F \) be the forgetful functor from the category of \( R \)-modules to the category of sets. Find a bijection from \( R \) to the set of natural transformations \( F \to F \). (Hint: use Yoneda.)

   **Solution:** The functor \( F \) is represented by \( R \) itself. Then Yoneda tells us that the set of natural transformations from \( F \) to \( R \) is the same as the set of homomorphisms \( R \to R \), which is just \( R \).

3. Let \( R \) be a ring and let \( e \) be an element of \( R \) such that \( e^2 = e \). Prove that the set \( D(e) \) of all prime ideals \( P \) of \( R \) which do not contain \( e \)
is closed in the Zariski topology of $R$. Conclude (and explain) that if $e \neq 0$ and $e \neq 1$, then Spec($R$) is not connected.

**Solution:** Let $e' := 1 - e$. Then $ee' = 0$ and $e + e' = 1$. Let $P$ be a prime ideal of $R$. The first of these equations implies that either $e$ or $e'$ belongs to $P$ and the second that only of them does. Thus Spec($R$) is the disjoint union of the two sets $D(e)$ and $D(e')$. Since both of these are open, they are also closed. If $e \neq 1$, then $e' \neq 0$, and since $ee' = 0$, it follows that $e$ is not a unit, and hence is contained some prime ideal. Thus $D(e')$ is not empty, and the same applies to $D(e)$. We have proved that Spec($R$) is the disjoint union of two nonempty open sets, and hence is disconnected.

4. Suppose that $R$ is a local ring and that $R/I$ is a flat $R$-module. Prove that either $I = 0$ or $I = R$. Note: Partial credit if you do this assuming that $I$ is finitely generated.

**Solution:** Let $J$ be a finitely generated ideal contained in $I$. By the flatness assumption, the map $J/IJ \to R/I$ is injective. Since it is also the zero map, it follows that $J = IJ$. If $I \neq R$, it is contained in the maximal ideal of $R$, and it follows from Nakayama’s lemma that $J = 0$. Since this is true for every finitely generated subideal of $I$, necessarily $I = 0$. 
