Homework Assignment #9:

April 1, 2016

Let A be a ring, let I be an ideal of A and let $\hat{A}$ denote the I-adic completion of A.

1. Show that if I is nilpotent, then the natural map $A \to A/I$ induces a bijection from the set of idempotents in A to the set of idempotents in A/I.
2. Show that the natural map from $\hat{A}$ to A/I induces a bijection from the set of idempotents of $\hat{A}$ to the set of idempotents of A/I.
3. More generally, let f be an element in the polynomial ring $A[x]$. Suppose that f and its derivative f' generate the unit ideal of $A/I[x]$. Show that the map $\hat{A} \to A/I$ induces a bijection on the set of zeroes of f.
4. Suppose now that A is noetherian and M a finitely generated A-module. Let $\hat{M}$ be the I-adic completion of M. Show that for every n, the natural map $M/I^nM \to \hat{M}/I^n\hat{M}$ is an isomorphism. Conclude that the natural map $\hat{M} \to \hat{\hat{M}}$ is an isomorphism.
5. Let A be the polynomial ring $k[x, y]$ in two variables over a field k and let I be the ideal (x, y).
   (a) Find an isomorphism from the Rees-algebra $B_I(R)$ to the A-algebra $A[X, Y]/(xY - yX)$.
   (b) Show that the I-adic completion $\hat{A}$ is isomorphic to the ring of formal power series $k[[x, y]]$.