Let $A$ be a ring, let $I$ be an ideal of $A$ and let $\hat{A}$ denote the $I$-adic completion of $A$.

1. Show that if $I$ is nilpotent, then the natural map $A \to A/I$ induces a bijection from the set of idempotents in $A$ to the set of idempotents in $A/I$.

2. Show that the natural map from $\hat{A}$ to $A/I$ induces a bijection from the set of idempotents of $\hat{A}$ to the set of idempotents of $A/I$.

3. More generally, let $f$ be an element in the polynomial ring $A[x]$. Suppose that $f$ and its derivative $f'$ generate the unit ideal of $A/I[x]$. Show that the map $\hat{A} \to A/I$ induces a bijection on the set of zeroes of $f$.

4. Suppose now that $A$ is noetherian and $M$ a finitely generated $A$-module. Let $\hat{M}$ be the $I$-adic completion of $M$. Show that for every $n$, the natural map $M/I^n M \to \hat{M}/I^n \hat{M}$ is an isomorphism. Conclude that the natural map $\hat{M} \to \hat{\hat{M}}$ is an isomorphism.

5. Let $A$ be the polynomial ring $k[x, y]$ in two variables over a field $k$ and let $I$ be the ideal $(x, y)$.

   (a) Find an isomorphism from the Rees-algebra $B_I(R)$ to the $A$-algebra $A[X, Y]/(xY - yX)$.

   (b) Show that the $I$-adic completion $\hat{A}$ is isomorphic to the ring of formal power series $k[[x, y]]$. 