Homework Assignment #7:

March 8, 2016

1. Let k be a field and let A be the localization of the polynomial ring $k[x, y]$ at the ideal (x, y), let B be the localization of the polynomial ring $k[x]$ at the zero ideal, and let $\theta \colon A \to B$ be the homomorphism sending y to zero. Then A and B are local rings, with maximal ideals m_A and m_B respectively. Show that $Tor_1^A(A/m_A, B) = 0$, but that B is not flat over A. Explain why this does not contradict the local criterion for flatness.

2. Let A be the polynomial ring $k[x, y]$, let B be the polynomial ring $k[s, t]$, and let $\theta \colon A \to B$ be the homomorphism sending x to s and y to st. Is this homomorphism flat? Which prime ideals of A can be lifted to B?

3. A ring homomorphism $\theta \colon A \to B$ is said to be "finite" if $\theta_*(B)$ is finitely generated as an A-module. Show that if this is the case and N is any finitely generated B-module, then $\theta_*(N)$ is also finitely generated as an A-module. Conclude that the composition of two finite ring homomorphisms is another finite ring homomorphism.

4. Show that if $A \to A'$ is any ring homomorphism and M is finitely generated as an A-module, then $A' \otimes M$ is finitely generated as an A'-module. Conclude that if $\theta \colon A \to B$ is finite, then $A' \to A' \otimes_A B$ is also finite. Thus the family of finite ring homomorphisms is stable under base change and, by the previous problem, under composition.

5. Let K be a field and let A be a K-algebra which is finite dimensional over K, and let $S := \mathrm{Spec}(A)$. Recall that every element Q of S is in fact a maximal ideal.

(a) Complete the proof sketched in class that S is finite and that the natural map

$$A \to \prod_{Q \in S} A/Q$$

is surjective. What is its kernel?

(b) Suppose that K is algebraically closed and that A is reduced. Let $X := h^A(K)$. Find a natural isomorphism of K-algebras: $A \to K^X$.

(c) Suppose that E/K is a finite separable field extension, and let X be the set of embeddings of E in some algebraic closure $\overline{K}$ of K. Find a natural isomorphism of $\overline{K}$-algebras: $\overline{K} \otimes_K E \to \overline{K}^X$. "Natural" includes in particular the statement that this isomorphism is compatible automorphisms of $\overline{K}$ over K. (Note how these automorphisms act on both sides.) Remarks: One definition of "separable field extension" is that the number of embeddings of E in $\overline{K}$ is equal to the degree of E over K. Another is that $\overline{K} \otimes_K E$ is reduced. Either of these definitions can be used to work this problem.