Homework Assignment #7:

March 8, 2016

1. Let $k$ be a field and let $A$ be the localization of the polynomial ring $k[x, y]$ at the ideal $(x, y)$, let $B$ be the localization of the polynomial ring $k[x]$ at the zero ideal, and let $\theta: A \to B$ be the homomorphism sending $y$ to zero. Then $A$ and $B$ are local rings, with maximal ideals $m_A$ and $m_B$ respectively. Show that $\text{Tor}^A_1(A/m_A, B) = 0$, but that $B$ is not flat over $A$. Explain why this does not contradict the local criterion for flatness.

2. Let $A$ be the polynomial ring $k[x, y]$, let $B$ be the polynomial ring $k[s, t]$, and let $\theta: A \to B$ be the homomorphism sending $x$ to $s$ and $y$ to $st$. Is this homomorphism flat? Which prime ideals of $A$ can be lifted to $B$?

3. A ring homomorphism $\theta: A \to B$ is said to be “finite” if $\theta_*(B)$ is finitely generated as an $A$-module. Show that if this is the case and $N$ is any finitely generated $B$-module, then $\theta_*(N)$ is also finitely generated as an $A$-module. Conclude that the composition of two finite ring homomorphisms is another finite ring homomorphism.

4. Show that if $A \to A'$ is any ring homomorphism and $M$ is finitely generated as an $A$-module, then $A' \otimes A M$ is finitely generated as an $A'$-module. Conclude that if $\theta: A \to B$ is finite, then $A' \to A' \otimes_A B$ is also finite. Thus the family of finite ring homomorphisms is stable under base change and, by the previous problem, under composition.

5. Let $K$ be a field and let $A$ be a $K$-algebra which is finite dimensional over $K$, and let $S := \text{Spec}(A)$. Recall that every element $Q$ of $S$ is in fact a maximal ideal.
(a) Complete the proof sketched in class that $S$ is finite and that the natural map
\[ A \to \prod_{Q \in S} A/Q \]
is surjective. What is its kernel?

(b) Suppose that $K$ is algebraically closed and that $A$ is reduced. Let $X := h^A(K)$. Find a natural isomorphism of $K$-algebras: $A \to K^X$.

(c) Suppose that $E/K$ is a finite separable field extension, and let $X$ be the set of embeddings of $E$ in some algebraic closure $\overline{K}$ of $K$. Find a natural isomorphism of $\overline{K}$-algebras: $\overline{K} \otimes_K E \to \overline{K}^X$. “Natural” includes in particular the statement that this isomorphism is compatible automorphisms of $\overline{K}$ over $K$. (Note how these automorphisms act on both sides.) Remarks: One definition of “separable field extension” is that the number of embeddings of $E$ in $\overline{K}$ is equal to the degree of $E$ over $K$. Another is that $\overline{K} \otimes_K E$ is reduced. Either of these definitions can be used to work this problem.