Homework Assignment #6:

March 3, 2016

1. Let $R$ be a commutative ring with identity.
   
   (a) Let $Z$ be a subset of $\text{Spec}(R)$ and let $I$ be the intersection of all the prime ideals of $R$ which belong to $Z$. Show that $\sqrt{I} = I$, where $\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n\}$. Show that $Z(I)$ is the closure of $Z$ in the Zariski topology.
   
   (b) Suppose that $Z_1$ and $Z_2$ are two closed subsets of $\text{Spec} R$, and let $I_i$ be the intersection of the elements of $Z_i$ (as above). Show that $Z_1$ and $Z_2$ are disjoint iff $I_1 + I_2 = R$, and show that $Z_1 \cup Z_2 = \text{Spec}(R)$ iff $I_1 \cap I_2 = \sqrt{0}$.
   
   (c) Suppose that $\text{Spec}(R)$ is disconnected. Show that then $R$ contains a nontrivial idempotent element. (Hint: Use the previous result to find $a, b \in R$ such that $1 = a + b$ and such that $ab$ is nilpotent. Then take a large power of both sides.

2. Let $M$ be an $R$-module and $I$ a nilpotent ideal (that is, $I^n = 0$ for some $n$). Show that if $M \otimes R/I = 0$, then $M = 0$.

3. Let $A \to B$ be a flat and local homomorphism of local rings. Suppose that $I$ is an nilpotent ideal of $A$ such that the induced map $A/I \to B/IB$ is an isomorphism. Prove that $A \to B$ is an isomorphism.

4. (Taken partly from Lang X, §3). Let $k$ be a field, let $R$ be the polynomial ring $k[t]$, let $B$ be the polynomial ring $R[x]$, and let $A$ be the subring of $B$ consisting of those polynomials $f = r_0 + r_1 x + r_2 x^2 + \cdots$ such that $r_1$ is divisible by $t$. Let $P := A \cap (x)B$, a prime ideal of $A$, and let $M$ be the $A$-module $A/P^2$. Find the associated primes of
Conclude that $P^2$ is not a primary ideal of $A$, i.e., that $M$ is not coprimary. Write $P^2$ as an intersection of two primary ideals in $A$. 
