Homework Assignment #3:
More on Limits

February 3, 2016

1. Let $\mathcal{C}$ be a category. Suppose that in $\mathcal{C}$, all products (limits over small discrete categories) and all equalizers are representable. Then all limits are representable. What is the dual statement?

2. Suppose that the category I has final object ∞. Show that if $C\cdot$ is any object of C^I, then the functor $\mathbf{colim}\, C\cdot$ taking T to $Mor_{C^I}(C\cdot, K(T))$ is representable by a pair (C_∞, ξ). What is ξ ? What is the dual statement?

3. Let $\mathcal{C}$ be a category. Prove that $\hat{\mathcal{C}}$ always has a final object. What is it?

4. Let F be the forgetful functor from the category of abelian groups to the category of sets. Show that F commutes with filtering direct limits. (This fact was mentioned in class.)

5. Let I and J be small categories and let $C\cdot$ be a bifunctor from $I \times J$ to a category $\mathcal{C}$. Prove that there is a natural transformation

$$\operatorname{colim} \lim \mathcal{C}\cdot \to \lim \operatorname{colim} \mathcal{C}\cdot,$$

   assuming both sides exist. Now assume that I is filtered and that J is finite (that is, it has only finitely many objects and only finitely many arrows). Prove that, if $\mathcal{C}$ is the category of sets, the above map is an isomorphism.

6. In the previous example, show that the map need not be an isomorphism if I and J are discrete and have two objects. Also show that

this map need not be an isomorphism if $I = J$ is the category with two objects $1, 2$, with two distinct arrows $a, b\colon 1 \to 2$. Hint. Try the following. Take $S_1, S_2 \in \mathcal{E}ns$ with $S_1 := \{x\}$, $S_2 := \{y, z\}$. Let $C_a\colon x \mapsto y$, $C_b\colon x \mapsto z$. Compute the equalizer and the coequalizer of C_a, C_b. Then build a suitable bifunctor.