Homework Assignment #12:

April 20, 2016

1. Let R be a regular local ring and let x be an element of the maximal ideal m of R. Give a necessary and sufficient condition on x for the quotient ring R/xR to be regular.

2. Let k be a field and let A be the polynomial ring $k[x, y, z]$, and let I be the ideal generated by $x, (1-x)y, (1-x)z$. Show that I is a proper ideal, that the sequence $(x, (1-x)y, (1-x)z)$ is A-regular, but that $((1-x)y, (1-x)z, x)$ is not A-regular.

3. Let I be an ideal in a noetherian ring A and let E be an A-module such that $E/IE \neq 0$. Recall that

$$\begin{aligned} depth_I E &:= \sup\{d : \exists\ E\text{-regular}(x_1, \ldots, x_d) \in I^d\} \\ &= \inf\{j : Ext^j(A/I, E) \neq 0\} \end{aligned}$$

   (a) Show that $depth_I E$ depends only on the radical of I.

   (b) If J is the annihilator of E, then E can also be viewed as an A/J-module. Show that the $depth_I E$ as an A-module is the same as $depth_{I/J \cap I} E$, as an A/J-module.

   (c) Show that if N is an A-module annihilated by I, then $Ext^i(N, E) = 0$ for $i < depth_I E$.

   (Hint: Use a filtration of N to reduce to the case in which N is monogenic.)

   (d) Show that

$$depth_I E = \inf\{depth E_P : P \in supp(E/IE)\},$$

   where here $depth E_P$ means the depth of E_P viewed as a module over the local ring A_P, with respect to its maximal ideal.

   (Hint: It is clear that $depth_I E \leq depth E_P$ for every $P \in supp(E/I)$, so we have $depth_I E \leq \inf\{depth E_P\}$. Prove the reverse inequality by inductionon the inf. Show that if this inf is positive then I is not contained in the union of the associated primes of E.)

4. Continuing with the hypothesis above, note that if $m < n$, the homomorphism $A/I^n \to A/I^m$ induces a homomorphism $Ext^i(A/I^m, E) \to Ext^i(A/I^n, E)$. Let $H_I^i(E) : \varinjlim Ext^i(A/I^n, E)$. Show that an exact sequence $0 \to E' \to E \to E'' \to 0$ induces a corresponding long exact sequece of H_I^*'s. Show that there is a natural injection $H_I^0(E) \to E$, and that an element of E lies in $H_I^0(E)$ if and only if it maps to zero in the localizaiton E_f for every $f \in I$.