1. Prove that the polynomial $y^6 - x^3 - 3$ has no zeroes in the ring $\mathbb{Z}$ of integers. (One possibility: reduce modulo a suitable prime $p$.)

2. Prove that the polynomial $y^2 + x^2 - 1$ is not a unit in the ring of polynomials with rational coefficients.

3. A groupoid is a category in which every morphism is an isomorphism. Prove that every groupoid $\mathcal{C}$ is isomorphic to its opposite $\mathcal{C}^o$, and in fact that there exists a “canonical” isomorphism $\mathcal{C} \rightarrow \mathcal{C}^o$. What does “canonical” mean?

4. Let $M$ be a monoid and let $S$ be a set. Define what is meant by a “left action of $M$ on $S$.” A set together with a left action of $M$ is called an “$M$-set.” Define what is meant by a morphism of $M$-sets. Let $\mathcal{B}_M$ be the category of $M$-sets, and let $F$ be the forget functor from $\mathcal{B}_M$ to the category of sets. Find all the natural transformations from $F$ to itself.

5. (You may find this problem too vague, since it contains some terms not defined in class. If so, just skip it.) Formulate elementary linear algebra as a statement about an equivalence of categories: $F: \mathcal{C} \rightarrow \mathcal{V}$. Here $\mathcal{V}$ is the category of finite dimensional vector spaces over a field and $\mathcal{C}$ is a category built from matrices. In $\mathcal{C}$, there should be only one element in each isomorphism class.

6. Let $R$ be a commutative ring and let $\mathcal{A}_R$ denote the category of $R$-algebras. Let $A$ be an object of $\mathcal{A}_R$, let $h^A$ be the functor from $\mathcal{A}_R$ to the category of sets taking $B$ to $\text{Mor}_{\mathcal{A}_R}(A, B)$, and let $F$ be the
forgetful functor from $A_R$ to the category of sets. Find all natural transformations from $h^A$ to $F$. 