Homework Assignment #1: Equations and Categories

January 22, 2016

1. Prove that the polynomial $y^6 - x^3 - 3$ has no zeroes in the ring $\mathbf{Z}$ of integers. (One possibiity: reduce modulo a suitable prime p.)

2. Prove that the polynomial $y^2 + x^2 - 1$ is not a unit in the ring of polynomials with rational coefficients.

3. A *grouipoid* is a category in which every morphism is an isomorphism. Prove that every groupoid $\mathcal{C}$ is isomorphic to its opposite $\mathcal{C}^o$, and in fact that there exists a "canonical" isomorphism $\mathcal{C} \to \mathcal{C}^o$. What does "canonical" mean?

4. Let M be a monoid and let S be a set. Define what is meant by a "left action of M on S." A set together with a left action of M is called an "M-set." Define what is meant by a morphism of M-sets. Let $\mathcal{B}_M$ be the category of M-sets, and let F be the forget functor from $\mathcal{B}_M$ to the category of sets. Find all the natural transformations from F to itself.

5. (You may find this problem too vague, since it contains some terms not defined in class. If so, just skip it.) Formulate elementary linear algebra as a statement about an equivalence of categories: $F \colon C \to V$. Here V is the category of finite dimensional vector spaces over a field and C is a category built from matrices. In C, there should be only one element in each isomorphism class.

6. Let R be a commutative ring and let $\mathcal{A}_R$ denote the category of R-algebras. Let A be an object of $\mathcal{A}_R$, let h^A be the functor from $\mathcal{A}_R$ to the category of sets taking B to $Mor_{\mathcal{A}_R}(A, B)$, and let F be the

forgetful functor from $\mathcal{A}_R$ to the category of sets. Find all natural transformations from h^A to F.