Sets and Correspondences

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Let A and B be sets. A *relation* from A to B is a subset R of $A \times B$. It would appear more natural to say a relation "between A and B", but since the roles of A and B are not exactly the same, this is not quite as precise. Note that the sets A and B cannot be determined from R, so that, strictly speaking, it doesn't make sense to say that A is the "domain" of R or that B is the "codomain" of R. To remedy this, we make the following more formal definition.

Definition A *correspondence* is a triple (A, B, R), where A and B are sets and R is a relation from A to B. The *domain* of a correspondence (A, B, R) is A, the *codomain* of (A, B, R) is B, and the *graph* of (A, B, R) is R. One says that $F := (A, B, R)$ is a correspondence from A to B, and writes symbolically:

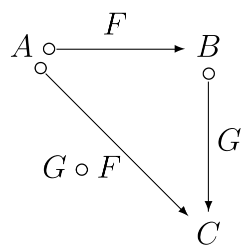
$$F \colon A \circ\!\!\longrightarrow B.$$

The relation R is called the *graph* of F and is often denoted by Γ_F.

If (A, B, R) is a correspondence from A to B and (B, C, S) is a correspondence from B to C, then one defines a new correspondence (A, C, T) from A to C by setting

$$T := S \circ R := \{(a, c) : \text{there exists some } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}.$$

This new correspondence is called the *composition* of F and G. If $F := (A, B, R)$ and $G := (B, C, S)$, then the composition of F and G is usually denoted by $G \circ F$ and by a diagram of the form:



This diagram is *commutative*, meaning that the correspondence from A to C is indeed the composite of the correspondence from A to B with the correspondence from B to C.

The *image* of a correspondence $F\colon A \circ\!\!\longrightarrow B$ is the set of $b \in B$ such that there exists an $a \in A$ such that $(a, b) \in \Gamma_F$. There is no commonly used term for the dual notion, that is the set of $a \in A$ such that there exists a $b \in B$ such that $(a, b) \in \Gamma_F$.

Examples and Exercises:

- If $f := (A, B, R)\colon A \circ\!\!\longrightarrow B$, then f^t (sometimes written f^{-1}) is (B, A, R^t), where $R^t := \{(b, a) : (a, b) \in R\}$.
- Show that $f^{t^t} = f$ and that $(f \circ g)^t = g^t \circ f^t$.
- The empty correspondence $e_{A,B}$ from A to B is $(A, B, \emptyset)$.
- The full correspondence from $f_{A,B}$ A to B is $(A, B, A \times B)$.
- The identity correspondence id_A from A to A is (A, A, Δ_A), where $\Delta_A := \{(a, a) : a \in A\}$ (the diagonal or identity relation on A).
- Show that if f, g, and h are correspondences such that $g \circ f$ and $h \circ g$ are defined, then $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are both defined and are equal.
- Show that if $f\colon A \circ\!\!\longrightarrow B$, then $\mathrm{id}_B \circ f = f$ and that $f \circ \mathrm{id}_A = f$.
- If $f\colon A \circ\!\!\longrightarrow B$, compute $e_{B,C} \circ f$ and $f_{B,C} \circ f$, and similarly on the other side. (Note: see a few lines above for the notation.)

Terminology:

- A correspondence $F\colon A \circ\!\!\longrightarrow B$ is called a *function* if for every $a \in A$, there is exactly one $b \in \Gamma_F$ such that $(a, b) \in \Gamma_F$. In this case one writes $F(a)$ for b and $F\colon A \longrightarrow B$ instead of $F\colon \circ\!\!\longrightarrow B$.
- A relation $R \subset A \times A$ is called *transitive* if $R \circ R \subseteq R$.
- A relation $R \subseteq A \times A$ is called a *preorder* if it is transitive and $\Delta_A \subseteq R$.
- A relation $R \subseteq A \times A$ is called an equivalence relation if it is a preorder and also $F = F^t$.
- A partition of a set A is a set Π of nonempty subsets of A such that each element of A is contained in exactly one element of P. Review the fact that there is a bijection between partitions of A and equivalence relations on A.