1. Lang V, 20.

   (a) Let F be a field and let $E = F(x)$, where x is transcendental over F. Let $K \neq F$ be a subfield of E. Then x is algebraic over K.

   Proof: Suppose that y belongs to K but not to F. Then y can be written as $f(x)/g(x)$, where f and g are monic polynomials with coefficients in F, and at least one of them has positive degree. Write $f = \sum a_i x^i$ and $g = \sum b_i x^i$, Then $y = f(x)/g(x)$, so $f(x) = yg(x)$. Let $c_i := a_i - yb_i$. Then $\sum_i c_i x^i = 0$, and $\sum 1 c_i t^i$ is a nonzero polynomial with coefficients in K. This shows that x is algebraic over K. □

   (b) Suppose that in the context of the proof above, f and g are relative prime and let n be the maximum of the degrees of f and g. Then the degree of x over K is n.

   Proof: We have seen that $p(t) := \sum_i t^i \in K[t]$ annihilates x. Since this polynomial has degree n, the degree of x over K is at most n. To prove equality we must show that $p(t)$ is irreducible in $K[t]$. Note first that y is transcendantal over F, since otherwise x would be algebraic over F. Let $R := F[y]$, which is a unique factorization domain, and note that $p(t) =\in R[t]$. Furthermore the coefficient $c_i = a_i - b_i y$ is either zero (if $a_i = b_i = 0$, a unit (if $y = 0$ and $a_i \neq 0$, or irreducible (if $b_i \neq 0$ in R. It is clear then the the Gauss content of $p(t)$ is 1, unless there exist $a, b, d_i \in F$ such that $a_i - b_i y = d_i(a - by)$ for all i. But then $f(x) = a \sum d_i x^i$ and $g(x) = b \sum d_i x^i$, contradicting the assumption that f and g are relatively prime. Thus $p(t)$ is primitive in $R[t]$, and it will suffice to prove that it is irreducible as an element of $R[t] = F[y, t]$, which is a polynomial ring in two variables. We have $p(t) = f(t) - yg(t)$. Now let Suppose that $p(t) = \alpha(t, y)\beta(ty)$, with $\alpha, \beta \in F[y, t]$. We have to show that α or β is a unit. Write $p(t) = f(t) - yg(t)$. Then

   $$f(t) - yg(t) = \alpha(t, y)\beta(t, y).$$

   The degree in y of the left side is just 1, so at most one of α and β involves y, and since β has degree at most 1 in y, we can write $\beta = \gamma(t) = y\delta(t)$ and

   $$f(t) - yg(t) = \alpha(t)(\gamma(t) - y\delta(t) = \alpha(t)\gamma(t) - y\alpha(t)\delta(t).$$

Then $f(t) = \alpha(t)\gamma(t)$ and $g(t) = -\alpha(t)\delta(t)$. Since f and g were relatively prime, it follows that α is a unit, and this shows that $p(t)$ is irreducible. □

Lang V, 24 Let k be a field of characteristic p and let u and t be algebraically independent variables over k.

(a) Prove that $k(u,t)$ has degree p^2 over $k(u,t)$.

Proof: First work with the polynomial rings. It is clear that we have a k basis for $k[u,t]$ consisting of the set of monomials $u^i t^j$, and similarly have a basis for $k[u^p, t^p]$ consistsing of the monomials in which all the coeffients are divisible by p. But any i can be written uniquely as $i = i' + r$ where i' is divisible by p and $0 \leq r < p$. This shows that $\{u^r t^s : 0 \leq r, s < p\}$ forms a basis for $k[u,t]$ when viewed as a module over $k[u^p, t^p]$. Now if one localizes by the set of all nonzero elements of $k[u^p, t^p]$, we still have a basis for the localization of $k[u,t]$ viewed as a module over the field $[(u^p, t^p)$. But this module is an integral finite dimensional over a field, hence a field, and hence it is all of of $k(u,t)$. □

(b) Show that there are infinitely many field extensions between $k(u^p, t^p)$ and $k[u,t]$.

Proof: We prove this assuming that k is infinite. In this case let E_c be the field extension of $k(u^p, t^p)$ generated by $u^p + ct^p$. It is easy to see that $E_a \neq E_b$ if $a \neq b$. □