1. Recall that every subgroup $A$ of $\mathbf{Z}$ contains a unique generator $n \in \mathbf{N}$, which if $A \neq \{0\}$, is a basis of $A$. This gives a "classification" of the subgroups of $\mathbf{Z}$. Generalize this to find a classification of subgroups of $\mathbf{Z}^n$ for $n > 1$. For example, if $n = 2$, show that every such $A$ has a unique basis of one of the following types:

   (a) $\beta = ((a,b),(0,d))$, where $a, d > 0$ and $0 \leq b < d$.

   (b) $\beta = ((a,b))$, where $a > 0$.

   (c) $\beta = ((0,b))$, where $b > 0$.

2. Let $A$ and be an $m \times n$ matrix with coefficients in $\mathbf{Z}$. Left multiplication $A$ defines a homomorphism of groups $T_A \colon \mathbf{Z}^n \to \mathbf{Z}^m$. Let $G_A$ be the cokernel of this map. Thus $A$ gives a presenation of the group $G_A$. Note that if there exist invertible matrices $B$ and $C$ such that $A' = BAC$. then $G_{A'}$ is isomorphic to $G_A$. As we saw in class, one can find $B$ and $C$ which are products of elementary matrices such that $A'$ is a diagonal matrix, with entries $(d_1, d_2, \cdots d_r)$, where $d_i$ divides $d_{i+1}$ for all $i$. Then $G_A$ is isomorphic to $\oplus_i \mathbf{Z}/d_i\mathbf{Z}$.

   Prove that if $G = \oplus \mathbf{Z}/d_i\mathbf{Z}$ and $G'$ is a quotient of $G$, then $G'$ is isomorphic to $\oplus_i \mathbf{Z}/d_i'\mathbf{Z}$, where each $d_i'$ divides $d_i$. Hint: It suffices to treat the case in which $G'$ is the quotient of $G$ by a cyclic subgroup generated by some $g$. In terms of a presentation, this means taking a diagonal matrix $(d_1, \ldots)$ as above, sticking on one additional column, and then doing elementary row and column operations to compute the new sequence $(d_1', \ldots)$.

   You can use this to solve problem 43.