Lecture 4—Inverse functions

January 22

Definition Let $f: A \to B$ be a function.

1. f is *surjective* (or *onto*) if for every $b \in B$, there is at least one $a \in A$ such that $f(a) = b$.
2. f is *injective* (or *one-to-one*) if $f(a) = f(a')$ implies $a = a'$.

Think about trying to solve the equation $f(x) = y$: given y, find x such that $f(x) = y$. There are two issues:

1. Is there always at least one solution? If so, f is surjective.
2. Is there always at most one solution? If so f is injective.

In the best of all possible worlds, f is both injective and surjective. In this case there is a *function* which provides the solution.

Definition Let $f: A \to B$ and $g: B \to A$ be functions. Then g *is inverse to* f if both the following hold:

1. For every $b \in B$, $f(g(b)) = b$.
2. For every $a \in A$, $g(f(a)) = a$.

Theorem Let $f: A \to B$ be a function. Then f has an inverse g if and only if f is injective and surjective. Moreover, in this case g is unique.

If f is injective and surjective, it has a unique inverse g, and it is safe to write f^{-1} for this function.