Lecture 3—Exponentials and Logarithms

January 22

Review

Definition: Let A and B be sets. A *function from A to B* is a rule which assigns to each element a of A a specific element of B, often written $f(a)$.

Last time we discussed *power functions*. Given any real number r, there is a function

$$p_r\colon \mathbf{R}^+ \to \mathbf{R}^+ : x \mapsto x^r.$$

Here the domain and codomain are both the set $\mathbf{R}^+$ of positive real niumbers. (For some values of r, the domain and codomain can be made bigger.) Recall the rules:

$$x^{a+b} = x^a x^b, (x^a)^b = x^{ab}$$

Exponentials Functions

Now we change our point of view to define for each *positive* real number a, a function

$$\exp_a\colon \mathbf{R} \to \mathbf{R}^+ : x \mapsto a^x.$$

Don't confuse this with the power function above. Although the actual computation is in some sense the same, the function is different, since the input (variable) is the exponent, not the base.

Various values of a give different graphs:

If $a > 1$, the function $\exp_a(x)$ is *increasing.*

If $a < 1$, it is *decreasing.*

If $a = 1$ it is constant.

It is always true that $\exp_a 0 = 1$.

$\exp_{\frac{1}{a}}(x) = (\exp_a(x))^{-1} = \exp_a(-x)$.

Question: Why do we exclude $a = 0$ here?

Definition Let $f\colon A \to B$ be a function, where A and B are subsets of $\mathbf{R}$. Then f is

- *increasing* if $f(a) < f(a')$ whenever $a < a'$.
- *decreasing* if $f(a) > f(a')$ whenever $a < a'$.

The larger a is , the faster $\exp_a$ increases with x. To measure this, we write $y = \exp_a$. Recall that for fixed x and Δx, we measure the increase in y:

$$\begin{aligned} \Delta y &:= \exp_a(x + \Delta x) - \exp_a(x) \\ &= \exp_a(x)(\exp_a(\Delta x) - \exp_a(0)) \end{aligned}$$

Thus $\exp_a$ changes $\exp_a(x)$ times faster at x than it does at 0. This leads to the principal, which we will make more precise later:

Principle of exponential growth: The rate of change of $\exp_a$ is proportional to its value.

The constant of proportionality depends on a—it is bigger for bigger a and smaller for smaller a. For example, if $a = 1$, it is zero, since $\exp_a$ is constant.

Theorem-Definition There is exactly one number, called e, such that the slope of $\exp_e$ at $x = 0$ is 1. This number is approximately 2.718281828459045.

Logarithms

Theorem-Definition Let a and x be positive numbers. Then there is a unique real number y such that $x = a^y$. Then we define

$$\log_a\colon \mathbf{R}^+ \to \mathbf{R} : x \mapsto y, \quad \text{where } x = a^y.$$

Thus, the following equations are equivalent:

$$\begin{aligned} y &= \log_a x \\ x &= \exp_a y \end{aligned}$$

For example:

$$\begin{aligned} 1000 &= 10^3 \\ 1000 &= \exp_{10} 3 \\ 3 &= \log_{10} 1000 \end{aligned}$$

Why does this work? Why can we "undo" the exponential function?