Lecture 2—Functions

January 22

Review

Definition: Let A and B be sets. A *function from A to B* is a rule which assigns to each element a of A a specific element of B, often written $f(a)$.

Notation and terminology:

- We usually denote a function by a small letter, such as f.
- We write $f\colon A \to B$ to mean that f is a function from A to B.
- We write $a \mapsto b$ if $b = f(a)$.
- If $f\colon A \to B$, the set A is called the *domain* of f and the set B is called the *codomain* of f. These two sets should be specified as part of the definition of the function f.
- The *range* of a function $f\colon A \to B$ is the set of all elements b of B for which there exists some $a \in A$ such that $b = f(a)$. This is a *subset* of B in general. Figuring out what this subset is can be difficult.

Sometimes one is given a formula, with or without a description of some context, and asked to figure out the domain of a function on which the formula makes sense in the given context. For example, the formula $\sqrt{x^2 - 9}$ can be used to define a function whose domain is $[-3, 3] := \{x : -3 \le x \le 3\}$. However in some contexts, one may want to restrict to a smaller domain, for example, to $[0, 3] := \{x : 0 \le x \le 3\}$.

Examples

- Linear functions. Given real numbers m and b one can define a function

$$f\colon \mathbf{R} \to \mathbf{R} : x \mapsto mx + b.$$

The graph of this as a subset of $\mathbf{R} \times \mathbf{R}$ looks like a straight line. Then $b = f(0)$ is the "y-intercept" of the line and m is the "slope." Review the meaning of this: Given any $x_0 \in \mathbf{R}$, if we compute $b_0 := f(x_0)$, then m

will tell us how much f will change if we change x. Precisely, if Δx is the amount we change x_0, so that $x_1 := x_0 + \Delta x$, then f changes by $m\Delta x$. Precisely, if $\Delta y := f(x_1) - f(x_0)$, then

$$\Delta y = m\Delta x.$$

What makes the graph a straight line is the fact that this equation holds for all x_0 and all Δx.

- Power functions. Given a natural number n (that is, $n = 0, 1, 2, \ldots$), we can form the function

  $$f_n\colon \mathbf{R} \to \mathbf{R} : x \mapsto x^n.$$

  You should know what the graphs of these look like. Note that

  $$\begin{aligned} f_n(-x) &= f_n(x) \quad \text{if } n \text{ is even} \\ f_n(-x) &= -f_n(x) \quad \text{if } n \text{ is odd} \end{aligned}$$

  For negative n, $f_n(x)$ is only defined if $x \neq 0$:

  $$f_n\colon \mathbf{R} \setminus \{0\} \to \mathbf{R}.$$

  If r is a rational number, that is, $r = p/q$ where p and q are integers (with $q \neq 0$), one can define f_r by doing some work as follows.

  Theorem-Definition Let x be a nonnegative real number and let $r := p/q$ be a rational number. Then there is a unique real number y such that $y^q = x^p$. By definition, $f_r\colon \mathbf{R}_{\geq} \to \mathbf{R}_{\geq}$ is the function which takes each x to the corresponding y.

  In fact, this definition can be extended to work for irrational values of r as well, but it takes some time to carry this out. (One must approximate such values of r by rational ones.)

New functions from old

Definition Let f and g buy two functions from a set A to $\mathbf{R}$. Then

$$\begin{aligned} (f+g)\colon A \to \mathbf{R} \quad &: \quad a \mapsto f(a) + g(a) \\ (fg)\colon A \to \mathbf{R} \quad &: \quad a \mapsto f(a)g(a) \end{aligned}$$

$f + g$ is called the *sum* of f and g and fg is called the *product* of f and g. If f is constant, one can usually figure out the graph of $f + g$ and of fg from the graph of f pretty easily, but in general it takes some thought.

The most important way to combine functions in general is by*composition.*

Definition Let A, B, and C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions, Then $g \circ f$ is the function $A \to C$ defined by:

$$g \circ f\colon A \to C : a \mapsto g(f(a)).$$