Continuity and the Intermediate Value Theorem

January 22

Theorem: (The Intermediate Value Theorem) Let a and b be real numbers with $a < b$, and let f be a real-valued and continuous function whose domain contains the closed interval $[a, b]$. Suppose that y is a real number between $f(a)$ and $f(b)$. Then there is some x in the interval $[a, b]$ such that $f(x) = y$.

This is a deep theorem whose proof requires the background on real numbers studied in Math 104.

Here is a partial converse to the theorem.

Theorem: Let a and b be real numbers with $a < b$ and let f be a real-valued function whose domain is $[a, b]$. Suppose also that f is increasing, that is, that $f(x) < f(x')$ whenever $x < x'$, and suppose also that the range of f is the entire closed interval $[f(a), f(b)]$. Then f is continuous. There is an analogous result if f is decreasing.

Proof: Let x_0 be a point of $[a, b]$ and let $y_0 := f(x_0)$. If x_0 belongs to the open interval (a, b), we have to prove that

$$\lim_{x \to x_0} f(x) = y_0;$$

if $x_0 = b$ we just look at the limit from below, and if $x_0 = a$, we just look at the limit from above, and if $x \in (a, b)$, we have to look at both. Since the basic idea is the same in all cases, I will here just look at the case when $x_0 < b$ and prove that the limit from above is y_0. Remember what this means:

For every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - y_0| < \epsilon$ whenever $x_0 < x < x_0 + \delta$.

This is suprisingly easy. Given $\epsilon > 0$, let y_1 be the minimum (smaller) of $y_0 + \epsilon$ and $f(b)$. Since $x_0 < b$, $y_0 := f(x_0) < f(b)$. Since y_1 is either $y_0 + \epsilon$ or $f(b)$, we can be sure that $y_0 < y_1$. Since $a \leq x_0$, we also know that $f(a) \leq y_0$. Thus y_1 belongs to the closed interval $[f(a), f(b)]$. By our assumption, every point in this interval belongs to the range of f, so there is some x_1 such that $f(x_1) = y_1$. Note that if it were the case that $x_1 \leq x_0$, then since f is increasing we would have $y_1 = f(x_1) \leq f(x_0) = y_0$, which is not so. Thus $x_0 < x_1$. Now let $\delta := x_1 - x_0$ which is positive. This δ works!. In fact, if x is any number such that $x_0 < x < \delta$, then just from the fact that f is increasing we can conclude that $y_0 := f(x_0) < f(x) < y_1$. But $y_1 \leq y_0 + \epsilon$, so then $|f(x) - y_0| < y_1 - y_0 \leq \epsilon$. $\square$

The intermediate value theorem can also be used to show that a continuous function on a closed interval $[a, b]$ is injective (one-to-one) if and only if either it is increasing or it is decreasing. Once one know this, then the inverse function must also be increasing (or decreasing), and it follows then from the theorem above that the inverse function is again continuous. See page A42 (Appendix F) of the textbook.