The Fundamental Theorem of Calculus

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The fundamental theorem of calculus has two parts:

Theorem (Part I). Let f be a continuous function on $[a, b]$ and define a function $g\colon [a, b] \to \mathbf{R}$ by

$$g(x) := \int_a^x f.$$

Then g is differentiable on (a, b), and for every $x \in (a, b)$,

$$g'(x) = f(x).$$

At the end points, g has a one-sided derivative, and the same formula holds. That is, the right-handed derivative of g at a is $f(a)$, and the left-handed derivative of f at b is $f(b)$.

Proof: This proof is surprisingly easy. It just uses the definition of derivatives and the following properties of the integral:

1. If f is continuous on $[a, b]$, then $\int_a^b f$ exists.
2. If f is continous on $[a, b]$ and $c \in [a, b]$, then

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

3. If $m \leq f \leq M$ on $[a, b]$, then

$$(b - a)m \leq \int_a^b f \leq (b - a)M.$$

Let x be a point in (a, b). (We just treat the case of $x \in (a, b)$ since the endpoints can be treated similarly.) If $x \in (a, b)$, we shall show that $g'(x^+) = g'(x^-) = f(x)$. Knowing that the two one-sided derivatives exist and are equal, we can conclude that the derivative exists and has this value.

By definition,

$$g'(x^+) = \lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h}.$$

Property (1) assures us that g is well defined provided that $h < b - x$. Property (2) allows us to simplify the numerator, since it implies that

$$g(x+h) - g(x) := \int_a^{x+h} f - \int_a^x f = \int_x^{x+h} f. \tag{1}$$

This is already great, since we only need to worry about f over the small interval $[x, x+h]$. A picture is helpful here, but I don't have time to include one in thise notes. Draw one yourself.

Now recall the definition of a limit. We have to show that given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| < \epsilon. \tag{2}$$

whenever $0 < h < \delta$. This is where we use the continuity of f at x. We know from this that there is a δ such that $|f(x') - f(x)| < \epsilon$ whenever $|x' - x| < \delta$. This means that

$$f(x) - \epsilon < f(x') < f(x) + \epsilon.$$

for all such x'. We use this same δ our criterion for the limit in equation (2). Let us verify that this works. Suppose that $0 < h < \delta$. Then on the interval $[x, x+h]$, we know that f is between $f(x) - \epsilon$ and $f(x) + \epsilon$. By property (3) of integrals, it follows that

$$(f(x) - \epsilon)h \leq \int_x^{x+h} f \leq (f(x) + \epsilon)h.$$

Since $h > 0$, we can divide both sides by h to conclude that

$$f(x) - \epsilon \leq 1/h \int_x^{x+h} f \leq f(x) + \epsilon, \quad i.e.,$$

$$f(x) - \epsilon \leq \frac{g(x+h) - g(x)}{h} \leq f(x) + \epsilon$$

This is exactly what we needed.

The left handed derivatives are done in essentially the same way.

□

Theorem (Part II). Let f be a continuous function on $[a, b]$. Suppose that F is continuous on $[a, b]$ and that $F' = f$ on (a, b). Then

$$\int_a^b f = F(b) - F(a).$$

Proof: Consider the function g in the previous theorem. Since g is differentiable on $[a, b]$ it is continuous there (including at the end points, where the one-sided deritaives exist). We also know that g and F are differentiable on (a, b), and that there derivatives are equal. Recall that we had (as a consequence of the mean value theorem for derivatives) that F and g differ by a constant. That is, there is a number C such that $g(x) = F(x)$ for all $x \in [a, b]$. Then

$$F(b) - F(a) = (g(b) + C) - (g(a) + C) = g(b) - g(a) = \int_a^b f - \int_a^a f = \int_a^b f.$$

□