The tangent approximation

March 10

Let f be a function of f. Suppose that we can calculate $f(a)$ and $f'(a)$. Then the tangent line to the graph of f at a is the line passing through the point $(a, f(a))$ whose slope is $f'(a)$. It is given by the formula

$$\ell_a(x) = f'(a)(x - a) + f(a).$$

It is a theorem that this is the line which best approximates f near a. Although we won't try to say exactly what this means, we will explain how well it does approximate f. The key point is that, for x near a, the difference between $f(x)$ and $\ell_a(x)$ is small *even compared to* $|x - a|$.

As a simple example, consider the function $f(x) = x^2$. Then $f'(a) = 2a$ and $\ell_a(x) = 2a(x - a) + a^2 = 2ax - a^2$. The point is, if we already know a^2, this is easier to compute than x^2 and is supposed to be near to x^2 if x is near a. How near? We can compute the difference:

$$|f(x) - \ell_a(x| = \left|x^2 - 2ax + a^2\right| = |x - a|^2$$

Note that if $|x - a| < 1$, this is small even compared to $|x - a|$.

Here is a precise statement.

Theorem: Suppose that $f'(a)$ exists. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - \ell_a(x)| \leq \epsilon |x - a|$$

whenever $|x - a| < \delta$.

For example, if ϵ is chosen to be .01, the error caused by using $\ell_a(x)$ in place of $f(x)$, will be at most 1% of the the difference between x and a.

Proof: From the definition of derivative:

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

By the definition of a limit, we can find $\delta > 0$ such that

$$\left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon$$

whenever $0 < |x - a| < \delta$. Now multiply both sides by the positive number $|x - a|$ to see that

$$|f(x) - f(a) - f'(a)(x - a)| < \epsilon|x - a|$$

whenever $0 < |x - a| < \delta$. Note that both sides vanish when $x = a$, so if we replace the "less than" sign by a "less than or equal" sign, the statements remains true for all x with $|x - a| < \delta$. Now if we substitute in the definition of $\ell_a(x)$, we see that

$$|f(x) - \ell_a(x)| \leq \epsilon|x - a|$$

whenever $|x - a| < \delta$. This proves the theorem.

Let's work out an example. Let $f(x) = \sqrt{x}$, for $x > 0$. Then if $a > 0$, $f'(a) = 1/2a^{-1/2}$, so $\ell_a(x) = (x - a)/2\sqrt{a} + \sqrt{a}$. Let's see if, given ϵ we can find δ that works in the above argument. In the calculations below, we shall frequently use our old friend: $A^2 - B^2 = (A + B)(A - B)$.

$$\begin{aligned}
|f(x) - \ell_a(x)| &= \left|\sqrt{x} - \sqrt{a} - \frac{(x - a)}{2\sqrt{a}}\right| \\
&= \left|\left(\sqrt{x} - \sqrt{a}\right)\left(1 - \frac{\sqrt{x} + \sqrt{a}}{2\sqrt{a}}\right)\right| \\
&= \left|\frac{x - a}{\sqrt{x} + \sqrt{a}}\left(\frac{\sqrt{a} - \sqrt{x}}{2\sqrt{a}}\right)\right| \\
&= \left|\frac{(x - a)^2}{(\sqrt{x} + \sqrt{a})^2 2\sqrt{a}}\right|
\end{aligned}$$

This is still pretty messy. We don't have to be very clever to get something useful and simple however. Since $\sqrt{x}$ is positive, if we omit it from the denominator we will get something bigger. So we conclude:

$$|f(x) - \ell_a(x)| \leq \left|\frac{(x - a)^2}{2(\sqrt{a})^3}\right|.$$

In our example, $a = 25$, and so $\sqrt{a} = 5$ and $\sqrt{a}^3 = 125$, so we get

$$|f(x) - \ell_{25}(x)| \leq \left|\frac{(x-25)^2}{250}\right| = |x-25|\left|\frac{x-25}{250}\right|$$

Conclusion: if we take $\delta := 250\epsilon$, then if $|x - 25| < \delta$,

$$|f(x) - \ell_a(x)| \leq \epsilon|x - a|.$$

Thus this ϵ is a bound for the *relative error*

$$\rho := \frac{|f(x) - \ell_a(x)|}{|x - a|}$$

(which makes sense only if $x \neq a$).

Let's look at some values. The estimates we just did predict that the relative error ρ is bounded by $.004|x - 25|$. Since it is impossible to actually write down $f(x)$ exactly, I have written $\tilde{f}(x)$ to indicate the approximation given by my calculator.

x	$x - a$	$\ell_a(x)$	$\tilde{f}(x)$	ρ	$.004\lvert x - a\rvert$
25	0	5	5		
26	1	5.1	5.099019513592784	0.980486407216 E-3	4 E-3
24	-1	4.9	4.898979485566356	01.020514433644 E-3	4 E-3
25.1	.1	5.01	5.009990019950139	0.9980049861 E-4	4 E-4
24.9	-.1	4.99	4.9899899799498590	1.0020050141 E-4	4 E-4
25.001	.001	5.0001	5.00009999900002	0.99998 E-6	4 E-6

I should mention another set of conventions that you will find in our book and often other places as well. To introduce it, we first write let $h := x - a$. Then our expression becomes

$$f(a + h) \sim f'(a)h + f(a),$$

or better:

$$f(a + h) - f(a) \sim f'(a)h$$

This is useful because h is small, and the expression displays clearly how our approximation depends on this small number h.

We could also write Δx in place of h. Note that a could be anything, and could even be regarded as a "variable." In fact to emphasize this, people tend to write x in place of a. Then our goal is to approximate $f(x + \Delta x) - f(x)$. Here is the standard definition, using the language of "differentials," in which we now write dx in place of Δx.

Definition: Suppose f is differentiable. For any x in the domain of f and any real number dx,

$$dy := f'(x)dx \text{ and } \Delta y := f(x + dx) - f(x)$$

Our theorem then says that dy is very near to Δy if dx is small. In fact, the difference between dy and Δy is small even compared to dx.