Mathematics 1A — Fall 2009 — Michael Christ
Lecture 6 (Wednesday 9/9/2009)
Definition of Limit

The main topics of today's lecture are (i)the definition of $\lim_{x\to a} f(x) = L$, and (ii) how to work directly with the definition to determine whether a linit exists, and to calculate it, if it does exist.

In practice, one uses whenever possible the rules which we learned in §2.3; scientists and engineers rarely need to go back to the definition itself, and run, do not walk, to the nearest door if your personal physician tries to use the definition of a limit on you ... But it is the formal definition which really tells us what the concept of limit means, not the vague versions which we discussed in §2.2.

- Definition of $\lim_{x\to a} f(x) = L$. See our text.
- Brief discussion of what ε, δ, "for every" and "there exists" mean.

  It is these last two phrases which make limits so tricky, especially the "for every". We are all accustomed to solving one math problem at a time, but in order to prove that a limit exists, we have to *solve infinitely many problems all at once* (there is a separate problem for each δ).
- **No bonuses.** There is no single "right answer" for δ; every δ which is small enough is a correct answer. There is no bonus or reward for finding the largest or prettiest value of δ.
- **Example 1:** $\lim_{x\to 2}(5x + 3) = 13$.

  Step 1: Write down $|f(x) - L|$ and try to find a useful upper bound, involving $|x - a|$. In this case, $|f(x) - L| = |(5x + 3) - 13| = |5x - 10| = 5|x - 2|$. Since $a = 2$, we have succeeded in involving $|x - a|$. (Step 1 varies from one problem of this type to another; the algebra depends on exactly what f is.)

  Step 2: Write down "Let $\varepsilon > 0$." Always good for partial credit. This means "Let's pretend that someone has challenged us with a very small number."

  Step 3: Write down "Let $\delta =$ ", and leave a blank space after the = sign. (Unless it's the world's simplest problem of this type, you won't be able to guess a good δ yet. Patience.) Then write "Let $0 < |x - a| < \delta$", but fill in the value of a. (Again, good for partial credit.)

  Step 4: Use algebra to combine your inequality for $|f(x) - L|$ with the inequality $|x - a| < \delta$ to find an upper bound for $|f(x) - L|$ which involves δ. Here $|f(x) - L| = 5|x - 2| < 5\delta$.

Step 5: Now find a value for δ. Go back and write it into your blank space. In this example, in order to be sure that $|f(x)-L| < \varepsilon$, it suffices to have $\delta < \varepsilon/5$ since then $|f(x)-L| < 5\delta = 5(\varepsilon/5) = \varepsilon$ means $|f(x)-L| < \varepsilon$ whenever $|x-2| < \delta$.

Step 6: Declare victory.

- **Example 2:** Show that $\lim_{x\to 3} x^2 = 9$.

  Solution: Here $f(x) = x^2$, $a = 3$, and $L = 9$. $|f(x)-L| = |x^2-9| = |(x-3)(x+3)| \le |x-3| \cdot |x+3|$.

  Now we run into a new difficulty: The extra factor $|x+3|$ is not a bounded function of x, so we can't just say $|f(x)-L| \le 10000|x-3|$, say.

  We deal with this by deciding that whatever our δ is, it will not exceed 1. We get to choose δ, so that's perfectly legal. If $\delta \le 1$ then $|x-3| < \delta$ implies that $2 < x < 4$, so $|x+3| \le 7$.

  Thus: If $\delta \le 1$ and $|x-3| \le \delta$ then

  $$|x^2-9| \le 7|x-3|.$$

  Now the rest is easy: Let $\varepsilon > 0$. Define δ to be the *minimum* of $\varepsilon/7$ and 1. If $|x-3| < \delta$ then $|x^2-9| \le 7|x-3| < 7\delta \le \varepsilon/7$. Victory.

- **Example 3:** Show that $\lim_{x\to 1/4} \frac{1}{x} = 4$.

  Solution: Here $a = 1/4$, $f(x) = 1/x$, $L = 4$. If $x \neq 0$ then

  $$|f(x)-L| = |\frac{1}{x} - 4| = \frac{|-1+4x|}{|x|} = 4\frac{|x-\frac{1}{4}|}{|x|}.$$

  (Notice how putting the two terms over a common denominator brings out the desired factor of $x - \frac{1}{4}$.)

  Now we have a slightly worse version of the same difficulty which we encountered in Example 2: The factor $|x|^{-1}$ is not bounded above. But if $\delta \le 1/8$ then whenever $|x-1/4| < \delta$, $x \in (1/8, 3/8)$ so $|x|^{-1} < 8$. Therefore $|f(x)-L| < 4 \cdot 8|x-1/4| = 32|x-1/4|$ whenever $|x-1/4| < 1/8$.

  Define $\delta = \min(1/8, \varepsilon/64)$. Then whenever $0 < |x-1/4| < \delta$, $|f(x)-L| < 32|x-1/4| < 32\delta \le \varepsilon$. Victory.

- **Example 4:** Show that $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \frac{1}{2}$.

  Here $a = 1$, $L = \frac{1}{2}$, $f(x) = \frac{\sqrt{x}-1}{x-1}$.

  This example illustrates three new themes.

  (i) $f(a)$ need not be defined; here $a = 1$ is not in the definition of our function.

(ii) This is the limit of a ratio, of which both numerator and denominator have limiting values 0 as $x \to a$.
(iii) f is just plain complicated.

I did not have sufficient time to complete the discussion of this example in class; see below for a full solution.

Sometimes functions involving square roots can be simplified. Here is how to simplify this one: If $x > 0$ and $x \neq 1$ then

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{\sqrt{x}+1}{\sqrt{x}+1} \cdot \frac{\sqrt{x}-1}{x-1} \\ &= \frac{(\sqrt{x}+1)(\sqrt{x}-1)}{(\sqrt{x}+1)(x-1)} \\ &= \frac{x-1}{(\sqrt{x}+1)(x-1)} \\ &= \frac{1}{\sqrt{x}+1}.\end{aligned}$$

Solution: With this work out of the way, I'll show that $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \frac{1}{2}$. Here $a = 1$, $f(x) = \frac{\sqrt{x}-1}{x-1}$, and $L = \frac{1}{2}$. We have already calculated above that

$$f(x) - L = \frac{1}{\sqrt{x}+1} - \frac{1}{2}.$$

Alas, we need to work on this some more:

$$\frac{1}{\sqrt{x}+1} - \frac{1}{2} = \frac{2-(\sqrt{x}+1)}{2(\sqrt{x}+1)} = \tfrac{1}{2}\frac{1-\sqrt{x}}{\sqrt{x}+1}.$$

Double alas: the numerator is still awkward to work with, so I'll simplify it by multiplying on top and bottom by $1+\sqrt{x}$. We get:

$$f(x) - L = \frac{1-x}{2(1+\sqrt{x})^2}.$$

Pick a reasonable preliminary value of δ, say $\frac{1}{2}$. If $|x-1| < \frac{1}{2}$ then $\frac{1}{2} \leq x \leq \frac{3}{2}$, so because the function $(\sqrt{x}+1)^2$ is increasing, $(\sqrt{x}+1)^2 \geq (\sqrt{1/2}+1)^2$, so

$$\frac{1}{(\sqrt{x}+1)^2} \leq (\sqrt{1/2}+1)^{-2}, \quad \text{provided } |x-1| < \tfrac{1}{2}.$$

This number is too ugly to be seen in public, so let's give it a name: Define

$$A = (\sqrt{1/2} + 1)^{-2}.$$

We have learned that if $|x - 1| < \frac{1}{2}$, then $|f(x) - L| \leq \frac{1}{2}A|x - 1|$.

Now we start the proof proper: Let $\varepsilon > 0$. Define

$$\delta = \min\left(\tfrac{1}{2}, \varepsilon/A\right).$$

If $0 < |x - 1| < \delta$ then

$$|f(x) - \tfrac{1}{2}| < \tfrac{1}{2}A|x - 1| < A|x - 1| < A\varepsilon/A = \varepsilon.$$

Victory, with honor.