Math 1A — UCB, Fall 2009 — M. Christ
Solutions[1] to selecta from problem Set 8c

§4.3 # 8(a). The graph of the first derivative f' of a function f is shown [see p.295]. On what intervals is f increasing? Explain.
Solution. Since f has a derivative everywhere and the derivative never intersects the x-axis without crossing it, f is increasing exactly when its derivative is positive. This is the case on the intervals $(2, 4)$ and $(6, 9)$. □

§4.3 # 8(b). At what values of x does f have a local maximum or minimum? Explain.
Solution. f has a local maximum exactly where f' changes from positive to negative, and it has a local minimum exactly where f' changes from negative to positive. So f has a local maximum at $x = 4$ and a local minimum at each of $x = 2$ and $x = 6$. $x = 0$ is not a local maximum because it is an endpoint, so it's not contained in an open interval in the domain of f (see Definition 2 in 4.1). □

§4.3 # 8(c). On what intervals is f concave upward or concave downward? Explain.
Solution. This is really just a matter of looking at the graph. f is concave upward on $(0, 2)$, $(4, 6)$, and $(7\frac{1}{2}, 9)$. It's concave downward on $(2, 4)$ and $(6, 7\frac{1}{2})$. □

§4.3 # 8(d). What are the x-coordinates of the inflection points of f? Why?
Solution. We want those x-values where the concavity of f changes. These are $x = 2$, 4, 6, and $7\frac{1}{2}$. □

§4.3 # 16(a). Set $f(x) = x^2 \ln x$. Find the intervals where f is increasing or decreasing.
Solution. Note that the domain of f is $(0, \infty)$. The derivative of f is $2x \ln x + x = x(2 + \ln x)$. x is always positive, so f' is positive or negative exactly when $2 + \ln x$ is. And $2 + \ln x$ is negative on $(0, e^{-2})$ and positive on (e^{-2}, ∞), so these are the intervals on which f is decreasing and increasing, respectively. □

§4.3 # 16(b). Find the local maximum and minimum values of f.
Solution. f has exactly one critical number, namely $x = e^{-2}$. At this point, f' changes from negative to positive, so this is a local minimum. Plugging it into f gives that the only local extremum of f is a minimum at $(e^{-2}, -2e^{-4})$. □

§4.3 # 16(c). Find the intervals of concavity and the inflection points.
Solution. Taking the second derivative gives $f''(x) = 3 + \ln x$. This is negative on $(0, e^{-3})$ and positive on (e^{-3}, ∞). By the concavity test, these are the intervals on which f is concave downward and concave upward, respectively. f has one inflection point at $x = e^{-3}$. □

Solution. Graph at end. Note that f is not required to be continuous, so it is not necessary to attach the $y = a - x$ pieces as I did. □

Solution. Graph at end. As the mug fills up, the rate at which the depth is increasing increases until the mug is half full, then decreases. This corresponds to a graph that switches from concave up to concave down halfway through. The inflection point is the point at which the depth is increasing fastest, which is the point when the coffee is halfway up (at the thinnest part of the mug). □

§4.3 # 71. Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number c. Prove that f is increasing on the entire interval I.

Solution. f is differentiable on I, so by the increasing/decreasing text, f is increasing on the portion of I less than c and the portion of I greater than c. Moreover, f is continuous at c (since it is differentiable), so $f(c) = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$. Using ϵ's and δ's, one can show that if f is increasing on (a, c), then $\lim_{x\to c^-} f(x) > f(x)$ for any x in (a, c), and similarly if f is increasing on (c, b), then $\lim_{x\to c^+} f(x) < f(x)$ for any x in (c, b). So f is increasing at c, so it's increasing on I.

(In essence, the above statement about limits is that the limit from below of an increasing function is not going to "jump down", nor will the limit from above "jump up".) □

§4.3 # 73(a). If f and g are positive, increasing, concave upward functions on I, show that the product function fg is concave upward on I.

Solution. We want to show $(fg)''$ is nonnegative on I, given the assumptions that f and g are positive on I, and that f', f'', g', and g'' are nonnegative on I. Well,

$$\begin{aligned}(fg)'' &= (f'g)' + (fg')' \\ &= f''g + f'g' + f'g' + fg'' \\ &= f''g + fg'' + 2f'g' \\ &\geq 0\end{aligned}$$

since each term is nonnegative. □

§4.3 # 73(b). Show that part (a) remains true if f and g are both decreasing.

Solution. This changes the assumption from f', g' nonnegative to f', g' nonpositive. Thus, the product $f'g'$ is still nonnegative, so the conclusion still holds. □