Math 1A — UCB, Spring 2010 — A. Ogus
Solutions[1] for Problem Set 8b

§4.1 # 10. Sketch the graph of the function f that is continuous on $[1,5]$ and has no local maximum or minimum, but with 2 and 3 as critical numbers.
Solution. Critical numbers occur when $f'(c)$ does not exist or when $f'(c)=0$. If f is a function with corners (points of non-differentiability) at 2 and 3 and such that f only increases (so has no local minima or maxima), f will satisfy the problem. □

§4.1 # 33. Find the critical numbers of the function $s(t)=3t^4+4t^3-6t^2$.
Solution. To find the critical numbers, we calculate $s'(t)=12t^3+12t^2-12t$ and set $s'(t)=0$, so $12t^3+12t^2-12t=0$. Then $12t(t^2+t-1)=0$, so $t=0,(-1+\sqrt{5})/2,(-1-\sqrt{5})/2$. □

§4.1 # 63. If a and b are positive numbers, find the maximum value of $f(x)=x^a(1-x)^b$ for $0\le x\le 1$.
Solution. This is a function on a closed interval, so we need to check the critical numbers of $f(x)$ as well as 0 and 1. Calculate $f'(x)=ax^{a-1}(1-x)^b+-bx^a(1-x)^{b-1}$ and set $f'(x)=0$. Then $ax^{a-1}(1-x)^b=bx^a(1-x)^{b-1}$, so $a(1-x)=bx$ after dividing through, and so $a-ax=bx$, so $x=a/(a+b)$. We have $f(0)=0$, $f(1)=0$, and $f(a/(a+b))=(a/(a+b))^a(b/(a+b))^b=a^a/(a+b)^a(b^b/(a+b)^b)=a^ab^b/(a+b)^{a+b}$. This must be the maximal value. □

§4.2 # 18. Show that the equation $2x-1-\sin x=0$ has exactly one real root.
Solution. It's continuous everywhere, and its values at -500 and 500 are positive and negative, respectively, so the intermediate value theorem says that there is at least one root. Moreover, the derivative is $2-\cos x$, which is always positive, so the function is strictly increasing. So it has exactly one root. □

§4.2 # 23. If $f(1)=10$ and $f'(x)\ge 2$ for $1\le x\le 4$, how small can $f(4)$ possibly be?
Solution. The procedure here is essentially the same as in Example 5. □

§4.2 # 27. Show that $\sqrt{1+x}<1+\frac{1}{2}x$ if $x>0$.
Solution. We will show the equivalent inequality $f(x)=1+\frac{1}{2}x+\sqrt{1+x}>0$ for $x>0$. The derivative of f is $\frac{1}{2}+\frac{1}{2\sqrt{1+x}}$, which is positive for $x>0$ and nonnegative for $x\ge 0$.

Since $f(0)=0$, the argument in Example 5 implies that $f(x)\ge 0$ for $x\ge 0$. Furthermore, for $x>0$, the left hand side is strictly increasing, since its derivative is strictly positive. So, for any $x>0$, $f(x)>f(\frac{1}{2}x)\ge 0$. □

§4.2 # 33. Prove the identity

$$\arcsin\frac{x-1}{x+1}=2\arctan\sqrt{x}-\frac{\pi}{2}$$

Solution. In the language of Example 6, we want to prove $f(x)=\frac{\pi}{2}$, where

$$f(x)=2\arctan\sqrt{x}-\arcsin\frac{x-1}{x+1}$$

Since, evaluating at $x = 0$, the equation is clearly true, it is enough to show that $f'(x) = 0$ for all $x > 0$, since this is the domain of f. We calculate

$$\begin{aligned} f'(x) &= 2 \cdot \frac{1}{2\sqrt{x}} \cdot \frac{1}{1+\sqrt{x}^2} - \frac{d}{dx}\left(1 - \frac{2}{x+1}\right) \cdot \sqrt{\frac{1}{1-(\frac{x-1}{x+1})^2}} \\ &= \frac{1}{(1+x)\sqrt{x}} - \frac{2}{(x+1)^2} \cdot \frac{|x+1|}{2\sqrt{x}} \\ &= \frac{1}{(1+x)\sqrt{x}} - \frac{1}{|1+x|\sqrt{x}} \\ &= 0 \end{aligned}$$

where the absolute value sign is ignored since x is positive. □

§4.2 # 34. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.

Solution. By assumption, the car's speed is continuous and differentiable everywhere. This means we can apply the mean value theorem. Noting that acceleration is the derivative of speed, the theorem states that at some time between 2:00 and 2:10 (noninclusive), the car's acceleration was

$$\frac{50\text{ mi/h} - 30\text{ mi/h}}{10\text{ minutes}} = \frac{20\text{ mi/h}}{\frac{1}{6}\text{ h}} = 120\text{ mi/h}^2$$

□