Math 1A — UCB, Spring 2010 — A. Ogus
Solutions[1] for Problem Set 8

§3.7 #18. If a tank holds 5000 gallons of water which drains from the bottom of the tank in 40 minutes, the volume remaining in the tank after t minutes is $V = 5000(1 - \frac{t}{40})^2$.
Solution. The rate at which the water is draining is $V'(t) = 2 \cdot 5000(1 - \frac{t}{40})(-\frac{1}{40}) = -250(1 - \frac{t}{40})$.
(a) $V'(5) = -250(1 - \frac{1}{8}) = -\frac{7}{8}250$.
(b) $V'(10) = -250(1 - \frac{1}{4}) = -\frac{3}{4}250$.
(c) $V'(20) = -250(1 - \frac{1}{2}) = -\frac{1}{2}250$
(d) $V'(40) = -250(1 - \frac{40}{40}) = 0$
The water is coming out the fastest, when $|V'(t)|$ is largest, which is when $\frac{t}{40}$ is smallest. Thus we conclude that the water is coming out fastest at the beginning. The more time goes by the slower the water comes out, until it stops at time 40.

§3.7 #28. The frequency of vibrations of a vibrating violin string is given by $f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$.
Solution. (a) $\frac{df}{dL} = -\frac{1}{2}L^{-2}\sqrt{\frac{T}{\rho}}$.
$\frac{df}{dT} = \frac{1}{2L}\frac{1}{2\sqrt{\frac{T}{\rho}}}\frac{1}{\rho}$
$\frac{df}{d\rho} = -\frac{1}{2L}\frac{1}{2\sqrt{\frac{T}{\rho}}}\frac{T}{\rho^2}$
(b) The derivative $\frac{df}{dL}$ is negative, hence the graph $f(L)$ is decreasing. If we decrease the length of the string, the frequency and hence the pitch is higher.
The derivative $\frac{df}{dT}$ is positive, since L, ρ are positive. Hence the graph $f(T)$ is increasing. Thus, if the tension is increased, then also the pitch increases.
The derivative $\frac{df}{d\rho}$ is negative, hence the graph $f(\rho)$ is decreasing. Thus is the linear density is increased, the pitch decreases.

§3.7 #31. If $p(x)$ is the total value of the production when there are x workers in a plant, then the average productivity of the workforce at the plant is $A(x) = \frac{p(x)}{x}$.
Solution. $A'(x) = \frac{xp'(x)-p(x)}{x^2}$. $A'(x) > 0$ iff $\frac{xp'(x)-p(x)}{x^2} > 0$ iff $xp'(x) - p(x) > 0$ iff $p'(x) > \frac{p(x)}{x}$.
§3.8 # 3(a). A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420. Find an expression for the number of bacteria after t hours.
Solution. The problem tells us that if P is the population, then $dP/dt = kP$, where k is some fixed number. This is a differential equation with solution $P(t) = P_0e^{kt}$, where $P_0 = 100$, the initial population. Since we know $420 = P(1) = 100e^{k(1)}$, we see that $k = \ln(420/100)$, and so our answer is $P(t) = 100e^{\ln(420/100)t} = 100(4.2)^t$. □

§3.8 # 3(c)). Find the rate of growth after 3 hours.
Solution. The rate of growth is just dP/dt, and we already know $dP/dt = kP = \ln(4.2)P$. At $t = 3$ hours, we know (from part (a)) that $P(3) = 100(4.2)^3$, so $dP/dt = 100(4.2)^3\ln(4.2) \approx 10,632$ bacteria/hour. □

§3.8 # 8(a). Bismuth-210 has a half-life of 5.0 days. A sample originally has a mass of 800mg. Find a formula for the mass remaining after t days.

Solution. We know this is an example of exponential decay, so $m(t) = m_0 e^{kt}$, where $m_0 = 800$ mg, the staring mass, and k is a constant to be determined. Since we know the half-life is 5.0 days, $m(5) = 800e^{5k} = (1/2)m_0 = 400$, so $e^{5k} = 1/2$, and $5k = \ln(1/2)$, so $k = -(1/5)\ln(2)$. Then $m(t) = 800e^{-(1/5)\ln(2)t} = 8002^{-t/5}$. □

§3.8 # 8(b). Find the mass remaining after 30 days.

Solution. This is just $m(30) = 8002^{-30/5} = 12.5$mg. □

§3.8 # 8(c). When is the mass reduced to 1 mg?

Solution. We want to find the time t when $m(t) = 1$. So, $1 = 8002^{-t/5}$, then $\log_2(1/800) = -t/5$, so $t = -5\log_2(1/800) = 5\log_2(800) \approx 48.2$ days. □

§3.8 # 15(a). When a cold drink is taken from a refrigerator, its temperature is 5° C. After 25 minutes in a 20° C room its temperature has increased to 10° C. What is the temperature of the drunk after 50 minutes?

Solution. The surrounding temperature is 20° C, so if $T(t)$ is the temperature of the drink at time t, let $y(t) = T(t) - 20$. Then Newton's Law of Cooling says that $y(t) = y_0 e^{kt}$ for some constant k, where $y_0 = T(0) - 20 = 5 - 20 = -15$. We know $-15e^{k(25)} = y(25) = 10 - 20 = -10$, so $e^{k(25)} = 2/3$ and $25k = \ln(2/3)$, so $k = (1/25)\ln(2/3)$. Then finally $T(t) = y(t) + 20 = -15e^{(1/25)\ln(2/3)t} + 20 = 20 - 15(2/3)^{t/25}$. The problem asks for $T(50) = 20 - 15(2/3)^{50/25} = 13.3°$ C. □

§3.8 # 19(a). If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.

Solution. The formula for compounded interest gives the value at time t given by $A(t) = A_0(1+r/n)^{nt}$ where r is the rate, t is time (in years), and n is the number of times per year it is compounded. Here $A_0 = \$3000$ and $r = 0.05$, so: (i) $A(t) = 3000(1 + 0.05)^5 = 3828.84$, (ii) $A(t) = 3000(1 + 0.05/2)^{10} = 3840.25$, (iii) $A(t) = 3000(1 + 0.05/12)^{60} = 2850.08$, (iv) $A(t) = 3000(1 + 0.05/52)^{260} = 3851.61$, (v) $A(t) = 3000(1 + 0.05/365)^{1575} = 3852.01$, (vi) $A(t) = 3000e^{(0.05)5} = 3852.08$.

§3.9 # 13. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

Solution. Use similar triangles. Let d be the distance between the man and the pole and let x be the length of the man's shadow. Then $x/(d + x) = 6/15$, so $15x = 6x + 6d$ and $9x = 6d$. Then $x = (2/3)d$. The problem wants us to find $d(d + x)/dt = d(d)/dx + dx/dt$, since this is the speed of the tip of his shadow. We know $dx/dt = (2/3)d(d)/dt$ and we know $d(d)/dt = 5$ ft/s, from the problem. Then $d(d + x)/dt = (5/3)d(d)/dt = 25/3$ ft/s. □

§3.9 # 18(a). A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s. At what rate is his distance from second base decreasing when he is halfway to first base?

Solution. His distance from second base as a function of t is $d(t) = \sqrt{90^2 + (90 - 24t)^2}$ by the Pythagorean theorem. Then $d'(t) = (1/2)(8100 + (90 - 24t)^2)^{-1/2}(2(90 - 24t))(-24) = -24(90 - 24t)(8100 + (90 - 24t)^2)^{-1/2}$. He is halfway to first base when $24t = 90/2 = 45$, so when $t = 45/24 = 15/8$. Then $d'(15/8) = -10.7$ ft/s. □

§3.9 # 18(b). At what rate is his distance from third base increasing at the same moment?
Solution. His distance from second base as a function of t is $d(t) = \sqrt{90^2 + (24t)^2}$ by the Pythagorean theorem, again. So $d'(t) = (1/2)(8100 + (24t)^2)^{-1/2}(2(24t)(24))$ and $d'(15/8) = 10.7$ ft/s. Geometrically, this makes sense. He should be moving away from third base at the same speed he is moving toward first base. □

§3.9 # 24. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?
Solution. Let $V(t)$ denote the volume of water in the trough at time t. If the height of water at time t is $h(t)$, then the cross-sectional area of filled water is an isosceles triangle with height $h(t)$, and base $h(t)(3/1) = 3h(t)$ by similar triangles, so it has area $(3/2)h(t)^2$. Then $V(t) = 10(3/2)h(t)^2 = 15h(t)^2$. So $V'(t) = 30h(t)h'(t)$. If t_0 is the time when the water is 6 inches deep, $h(t_0) = 1/2$, since we measure al distances in ft. The problem also tells us that $V'(t_0) = 12$ ft^3/min. Then $12 = 30(1/2)h'(t_0)$, so $h'(t_0) = 12/15$ ft/min, and this is the rate at which the water level is rising when the water is 6 inches deep. □

§3.9 # 36. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?
Solution. Let $x(t)$ be the position of A with respect to Q and $y(t)$ be the position of B, where right is positive. The Pythagorean theorem tells us that $\sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2} = 39$, since the length of the rope is 39ft. We can differentiate this with respect to t and obtain $(1/2)(x^2 + 12^2)^{-1/2}(2xx') + (1/2)(y^2 + 12^2)^{-1/2}(2yy') = 0$. When cart A is 5 ft from Q, $x = -5$. and $x' = -2$. Furthermore, $y = \sqrt{532} \approx 23.1$ by solving for y above, so $y' = -(1/2)(x^2+12^2)^{-1/2}(2xx')/[y(y^2+12^2)^{-1/2}] = -0.87$ ft/s. □

§3.9 # 44. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?
Solution. Let $\theta(t)$ (t in hours) be the angle of the hour hand (so for 12 AM/PM, $\theta = \pi/2$ and $\psi(t)$ be the angle of the minute hand. The coordinates of the tip of the minute hand are $(8\cos\psi, 8\sin\psi)$ and the coordinates of the tip of the hour hand are $(4\cos\theta, 4\sin\theta)$, so the distance between them is $d = \sqrt{(8\cos\psi - 4\cos\theta)^2 + (8\sin\psi - 4\sin\theta)^2}$. Then $d(d)/dt = [(8\cos\psi - 4\cos\theta)^2 + (8\sin\psi - 4\sin\theta)^2]^{-1/2}[2(8\cos\psi - 4\cos\theta)(-8\sin\psi\psi' + 4\sin\theta\theta') + 2(8\sin\psi - 4\sin\theta)(8\cos\psi\psi' - 4\cos\theta\theta')$ and we know at 1 o'clock, $\theta = \pi/6$, $\psi = \pi/2$, $\theta' = 2\pi/12$, $\psi' = 2\pi/1$. We can substitute this in, so $d(d)/dt = 63.2$ mm/hr. □

§3.10 #2 Find the linearization of $f(x) = \ln x$ at $a = 1$.
Solution. $f'(x) = 1/x$, so $f'(a) = 1$. Since $f(a) = 0$, the linearization is given by $\ell_a(x) = x$.

§3.10 #34 The radius of a circular disk is given as 24 cm with a maximum error in measurement of .2 cm.
(a) Use differentials to estimate th maximum error in the calculated area of the disk.
Solution. The area is given by $A = \pi r^2$, and the differential $dA = 2\pi r dr$. So if $dr = .2$, $dA = 2\pi 24.2 = 9.6\pi$ square centimeters.
(b) The relative error is dA/A which is $2\pi r dr/\pi r^2 = 2dr/r = 2(.2)/24 = 1/60$, or .0167%.