Math 1A — UCB, Spring 2010 — A. Ogus
Solutions[1] for Problem Set 6

§3.1 # 2(b). What types of functions are e^x and x^e? Compare their derivatives.
Solution. e^x is an exponential function; x^e is a power function. Their derivatives are respectively e^x, and ex^{e-1}. □

§3.1 # 2(c). Which of e^x and x^e grows more rapidly as $x \to \infty$?
Solution. e^x grows far, far more rapidly. $e^{x+1} = e \cdot e^x > 2e^x$ for every x, so e^x more than doubles every time x passes from one integer n to $n+1$. In contrast, x^e doubles when x is multiplied by a factor r, where $r^e = 2$. (Then $(rx)^e = x^e \cdot r^e = 2$.) Clearly the former type of growth wins out in the long run. □

§3.1 # 35. Find the equation of the tangent and normal lines to the curve $y = x^4 + 2e^x$ at $(0, 2)$.
Solution. $\frac{dy}{dx} = 4x^3 + 2e^x$, so at $(0, 2)$, the slope of the tangent line is $4 \cdot 0^3 + 2e^0 = 2$. An equation for the tangent line is therefore $y = 2(x - 0) + 2 = 2x + 2$.

The normal line is the unique line through $(0, 2)$ which is perpendicular to the tangent lines. Two lines are perpendicular if and only if the product of their slopes is -1 (unless one is horizontal and the other, vertical). Thus in this case, the slope of the normal line is $-\frac{1}{2}$, so an equation for the normal is $y = -\frac{1}{2}(x - 0) + 2 = -\frac{1}{2}x + 2$. □

§3.1 # 50(a). The equation of motion of a particle is $s = 2t^3 - 7t^2 + 4t + 1$, where s is measured in meters and t in seconds. Find the velocity and acceleration as functions of t.
Solution. The velocity is $v = \frac{ds}{dt} = 6t^2 - 14t + 4$, and the acceleration is $a = \frac{dv}{dt} = 12t - 14$. □

§3.1 # 50(b). Find the acceleration after 1 second.
Solution. Plug in $t = 1$ to obtain $a = 12 - 14 = -2$. (The physical interpretation is that the particle is slowing down at this instant in time; its velocity is decreasing as time increases.) □

§3.1 # 51. Find all points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.
Solution. $\frac{dy}{dx} = 6x^2 + 6x - 12$. The tangent line is horizontal exactly where $\frac{dy}{dx} = 0$. We have $6x^2 + 6x - 12 = 6(x^2 + x - 1) = 6(x + 2)(x - 1)$, which vanishes if and only if $x = -2$ or $x = 1$. Thus there are two points with horizontal tangents: $(1, -6)$, and $(-2, 21)$.
§3.1 #62(a). Find the n-th derivative of $f(x) = x^n$ by calculating the first few derivatives, and observing the pattern.
Solution. $f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$, and $f^{(3)}(x) = n(n-1)(n-2)x^{n-3}$. Thus the k-th derivative of f is $f^{(k)}(x) = n(n-1)(n-2)\cdots(n-k+1)x^{n-k}$. For $k = n$, x is raised to the power 0, and we get

$$f^{(n)}(x) = n!;$$

$f^{(n)}$ is this constant function. □

§3.1 #62(b). Find the n-th derivative of $f(x) = 1/x$ by calculating the first few derivatives, and observing the pattern.
Solution. Write $f(x) = x^{-1}$. $f'(x) = -x^{-1}$, $f''(x) = +x^{-2}$, $f^{(3)}(x) = -2x^{-3}$, $f^{(4)}(x) = 2 \cdot 3 \cdot x^{-4}$, $f^{(5)}(x) = -2 \cdot 3 \cdot \cdot 4x^{-5}$. Thus

$$f^{(n)}(x) = (-1)^n 2 \cdot 3 \cdot \cdot 4 \cdots (n-1)x^{-n-1}. = (-1)^n (n-1)! x^{-n-1}.$$

□

§3.2 #18. Differentiate $y = \frac{1}{s+ke^s}$.
Solution. Although this is not stated, our author intends us to calculate the derivative of y with respect to s, not with respect to k, which often stands for a constant (or konstant) in various applications to physical science.

Using the quotient rule,

$$\frac{dy}{ds} = \frac{0 - \frac{d(s+ke^s)}{ds}}{(s+ke^s)^2} = -\frac{1+ke^s}{(s+ke^s)^2} = -(1+ke^s)(s+ke^s)^{-2}.$$

□

§3.2 #32. Find an equation for the line tangent to the curve $y = e^x/x$ at $(1, e)$.
Solution. Writing $y = x^{-1}e^x$, the derivative is

$$\frac{dy}{dx} = -x^{-2}e^x + x^{-1}e^x.$$

There are various ways to write the derivative; this one is fine for what we need to do next. Now plug in $x = 1$ to find that the slope of the tangent line at $(1, e)$ equals $-1 \cdot e^1 + 1 \cdot e^1 = 0$. Thus an equation of the tangent line is $y = 0(x-1) + e$, or more simply, $y = e$. □

§3.3 #10. Differentiate $y = \frac{1+\sin(x)}{x+\cos(x)}$.
Solution. $y = \frac{f(x)}{g(x)}$ where $f(x) = 1 + \sin(x)$ and $g(x) = x + \cos(x)$. Thus $f'(x) = \cos(x)$, while $g'(x) = x - \sin(x)$. Therefore

$$\frac{dy}{dx} = \frac{f'g - fg'}{g^2} = \frac{\cos(x)(x+\cos(x)) - (1+\sin(x))(x-\sin(x))}{(x+\cos(x))^2}$$

I see now way to dramatically simplify this, so I'll leave it in this form. □

§3.3 #24. Find an equation of the line tangent to the graph of $y = (\sin(x) + \cos(x))^{-1}$ at the point $(0, 1)$.
Solution.

$$\frac{dy}{dx} = -(\cos(x) - \sin(x)) \cdot (\sin(x) + \cos(x))^{-2} = (\sin(x) - \cos(x))(\sin(x) + \cos(x))^{-2}.$$

Plugging in $x = 0$ gives

$$\left.\frac{dy}{dx}\right|_{x=0} = (0-1)(0+1)^{-2} = -1.$$

An equation for the tangent line is

$$y = -1 \cdot (x - 0) + 1 = -x + 1.$$

□

§3.3 # 37. A ladder 10 feet long leans against a vertical wall. Let θ be the angle between the top of the ladder and the wall, and let x be the distance from the bottom of the ladder to the wall, also measured in feet. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?
Solution. (There is another variable in this problem, time. However, we are not asked how fast x or θ change *with respect to time*. Nor can we answer that question; nothing indicates whether the

ladder slides quickly (as it would if the floor were greased), or slowly (as it could if the floor were coated with tar).)

By examining the right triangle formed by the ladder, wall, and floor, we find that $\tan(\theta) = x/10$. Take the derivatives of both sides with respect to x:

$$\frac{1}{10}\frac{dx}{d\theta} = \frac{d}{d\theta}(\tan(\theta)) = \sec^2(\theta).$$

Thus $\frac{dx}{d\theta} = 10\sec^2(\theta)$. Since $\cos(\pi/3) = \frac{1}{2}$, $\sec^2(\pi/3) = 4$, so $\frac{dx}{d\theta} = 40$ when $\theta = \pi/3$. □

§3.3 # 51. (Please refer to figure in text, page 197.) The figure shows a circular arc of length s and a central chord of length d, both subtended by a central angle θ. Find $\lim_{\theta\to 0^+} \frac{s}{d}$.

Solution. Assume that the circle has radius 1. (If it does not, then we may scale the picture by a factor of one over the radius. This changes neither the ratio s/d, nor the angle θ.)

Now $\theta = s$, since the arc has length s and the circle has radius 1. (We always measure our angles in radians.) What is d? We could use the law of cosines; or we can simply observe that by drawing the line which passes through the center of the circle and bisects the chord, $\frac{d}{2} = \sin(\theta/2)$; $d = 2\sin(\theta/2)$. Thus

$$\lim_{\theta\to 0^+} \frac{s}{d} = \lim_{\theta\to 0^+} \frac{\theta}{2\sin(\theta/2)} = \lim_{\theta\to 0^+} \frac{\theta/2}{\sin(\theta/2)} = 1$$

(by substituting $t = \theta/2$; $t \to 0^+$ as $\theta \to 0^+$). □