Math 1A — UCB, Fall 2010 — A. Ogus
Solutions[1] for Problem Set 4

§2.5 # 22. Explain, using Theorems 4, 5, 7, and 9, why the function $\sqrt[3]{x}(1+x^3)$ is continuous at every member of its domain. State its domain.
Solution.
By Theorem 7, the function $\sqrt[3]{x}$ is defined and continuous on all of $\mathbb{R}$ because it is an odd root function (3 being an odd integer; see p. 101 theorem 10). By Theorem 7, the function $1+x^3$ is also defined and continuous on all of $\mathbb{R}$ because it is a polynomial. Hence by Theorem 4, the product of these functions $\sqrt[3]{x}(1+x^3)$ is defined and continuous on all of $\mathbb{R}$. (In particular its domain is $(-\infty, \infty)$.)

□

§2.6 # 27. Find the limit $\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2+bx}$
Solution.
Rationalize the difference of radicals:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2+bx} &= \lim_{x\to\infty} \left(\sqrt{x^2+ax} - \sqrt{x^2+bx}\right) \cdot \frac{\sqrt{x^2+ax} + \sqrt{x^2+bx}}{\sqrt{x^2+ax} + \sqrt{x^2+bx}} \\
&= \lim_{x\to\infty} \frac{(a-b)x}{\sqrt{x^2+ax} + \sqrt{x^2+bx}} \\
&= (a-b) \lim_{x\to\infty} \frac{x}{\sqrt{x^2+ax} + \sqrt{x^2+bx}} \cdot \frac{1/x}{1/x} \\
&= (a-b) \lim_{x\to\infty} \frac{1}{\sqrt{1+a/x} + \sqrt{1+b/x}} \\
&= (a-b) \frac{1}{\sqrt{1+0} + \sqrt{1+0)}} \\
&= \frac{a-b}{2}
\end{aligned}$$

□

§2.6 # 28. Find the limit $\lim_{x\to\infty} \cos x$
Solution.
This limit does not exist. Proof: Suppose the limit was some number L. Let $\varepsilon = 1/4$. Then no matter what N we take,
1) there will be some x-values $> N$ (namely, even multiples of π) such that $\cos x = 1$, and
2) there will be some x-values $> N$ (namely, odd multiples of π) such that $\cos x = -1$.
These outputs (± 1) do lie inside any interval of radius $\varepsilon = 1/4$ (i.e. an interval of length 1/2), so we could not possibly have $|\cos(x) - L| < \varepsilon$ for these outputs.

□

§2.6 # 48. Find a formula for a function which has vertical asymptotes $x = 1$ and $x = 3$, and horizontal asymptote $y = 1$

Solution.

For the vertical asymptotes, we'll have a denominator with factors of $x-1$ and $x-3$, and to make the limits at $\pm\infty$ equal to $y=1$, we should have the numerator of the same degree as the denominator, with the same coefficient. So, let

$$f(x) = \frac{x^2}{(x-1)(x-3)}$$

To be precise, we can check:

$$\lim_{x\to 1^-} f(x) = \infty,$$

$$\lim_{x\to 1^+} f(x) = -\infty,$$

$$\lim_{x\to 3^-} f(x) = -\infty$$

$$\lim_{x\to 3^+} f(x) = \infty$$

$$\lim_{x\to -\infty} f(x) = 1$$

$$\lim_{x\to \infty} f(x) = 1$$

□

§2.6 # 52. Find the limits of $f(x) = x^2(x^2-1)^2(x+2)$ as $x \to \infty$ and $\to -\infty$. Use this information, along with intercepts, to give a rough sketch of its graph as in Example 11.

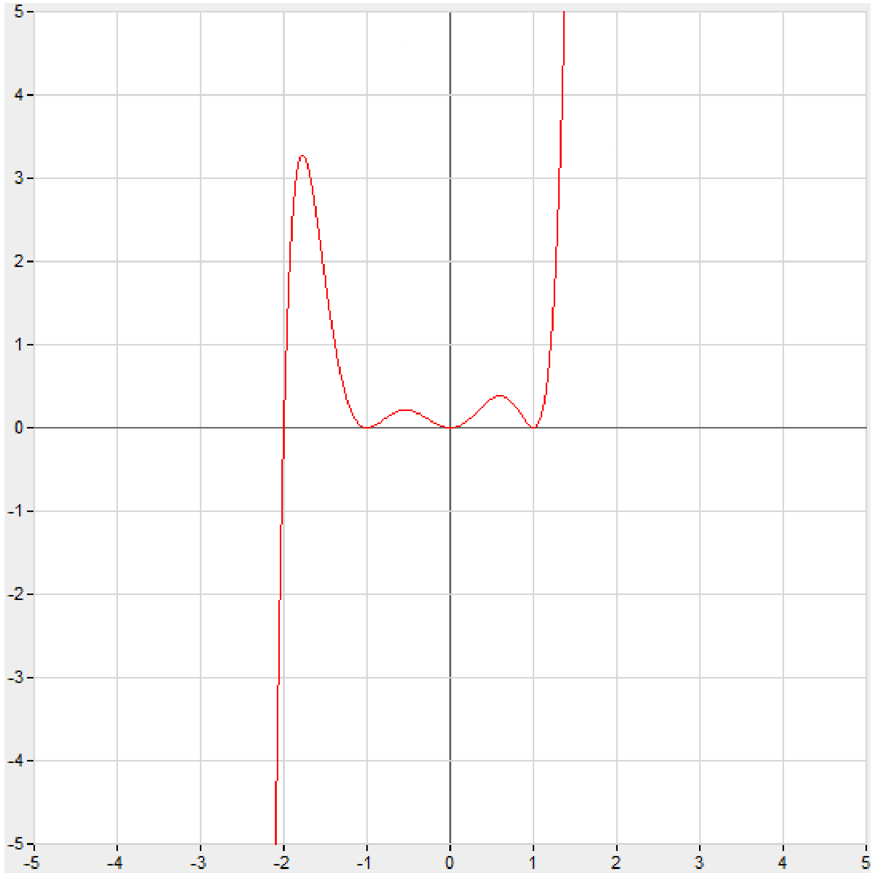
Solution. For x large positive, x^2, $(x^2-1)^2$, and $(x+2)$ are all large positive, so $f(x)$ is large positive:

$$\lim_{x\to\infty} x^2(x^2-1)^2(x+2) = \infty$$

For x large negative, x^2 and $(x^2-1)^2$ are large positive, and $(x+2)$ is large negative, so $f(x)$ is large negative:

$$\lim_{x\to-\infty} x^2(x^2-1)^2(x+2) = -\infty$$

The function $f(x) = x^2(x-1)^2(x+1)^2(x+2)$ has an x-intercept at $x=-2$, and "double" x-intercepts at each of $x=-1$, $x=0$, and $x=1$. It y-intercept is $y=0$, so the graph looks like:



§2.7 # 1. A curve has equation $y = f(x)$.

(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.

(b) Write and expression for the tangent line at P.

Solution.

(a) The slope of the secant line is $\frac{f(x)-f(3)}{x-3}$.

(b) The slope of the tangent line is $\lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$.

$\square$

§2.7 # 10.

(a) Find the slope of the tangent line to the curve $y = \frac{1}{\sqrt{x}}$ at the point where $x = a$.

(b) Find the equations of the tangent lines at the points $(1, 1)$ and $(4, \frac{1}{2}$

Solution.

(a) The slope is (first get rid of the fractions in the numerator, and the rationalize)

$$\begin{aligned}
\lim_{x\to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x-a} &= \lim_{x\to a} \frac{\sqrt{a}-\sqrt{x}}{(x-a)\sqrt{xa}} \\
&= \lim_{x\to a} \frac{a-x}{(x-a)\sqrt{xa}(\sqrt{a}+\sqrt{x})} \\
&= \lim_{x\to a} \frac{-1}{\sqrt{xa}(\sqrt{a}+\sqrt{x})} \\
&= \frac{-1}{\sqrt{a^2}(2\sqrt{a})} = \frac{-1}{2a\sqrt{a}}
\end{aligned}$$

(b) To find the slope of the tangent line at $(1,1)$, we plug in $a=1$ to the answer above, and find the slope is $\frac{-1}{2}$. Since the line must also pass through $(1,1)$, we find the equation is

$$y-1 = \frac{-1}{2}(x-1)$$

or more simply $y = \frac{-1}{2}x + \frac{3}{2}$.

Similarly, the tangent line through $(4, \frac{1}{2})$ has slope $\frac{-1}{16}$, by plugging in $a=4$, so the equation for this tangent line is $y - \frac{1}{2} = \frac{-1}{16}(x-4)$, which simplifies to $y = \frac{-1}{16}x + \frac{3}{4}$.

□

§2.7 # 12. Looking at the graph in the book,

(a) Describe and compare how the runners ran the race.

(b) At what time is the distance between the runners the greatest?

(c) At what time do they have the same velocity?

Solution.

(a) Since the first graph is a straight line, runner A ran at a constant speed. Runner B started out slower and then gradually sped up, and was running faster than A at the end. They both ran the same total distance in the same amount of time.

(b) The distance between the runners at any time is the vertical distance between their graphs, which looks to be the greatest at $t = 10$.

(c) They have the same velocity when their tangent lines have the same clope. Since A is a straight line, I can look for when B's tangent line is parallel to A, which is approximately at $t = 10$.

□

§2.7 # 17. For the function g whose graph is given (in the book), put the numbers in order.

Solution.

$$g'(0) < 0 < g'(4) < g'(2) < g'(-2)$$

Of the points we are looking at, $x=0$ is the only one where g is decreasing, so $g'(0)$ is negative and all the others are positive. g is increasing pretty slowly at $x=4$, so $g'(4)$ is smaller than the other two. $g'(2)$ and $g'(-2)$ are close, but it looks like g is increasing a little faster at $x=-2$, so g' is the largest there.

□

§2.7 # 20. Sketch a graph of a function g for which $g(0) = g'(0) = 0, g'(-1) = -1, g'(1) = 3$, and $g'(2) = 1$.

Solution.

Of course, many graphs will work, it should cross the origin and be flat there. It should be increasing very quickly at $x = 1$, and more slowly at $x = 2$ and $x = -1$.

□

§2.7 # 32. Find an f and a so that this expression is the derivative of f at a:

$$\lim_{h \to 0} \frac{\sqrt[4]{16+h} - 2}{h}$$

Solution.

$f(x) = \sqrt[4]{x}$ at $a = 16$.

□

§2.7 # 41. Use the table which shows the percentage of Europeans using cell phones in various years to:

(a) Find the average rate of cell phone growth from various years

(b) Estimate the instantaneous rate of growth in 2000 by taking the average of two rates of change. What are the units?

(c) Estimate the instantaneous rate of growth in 2000 by measuring the slope of a tangent.

Solution.

(a) from 2000 to 2002: $\frac{77-55}{2002-2000} = 11$ percentage points per year. from 2000 to 2001: $\frac{68-55}{2001-2000} = 13$ percentage points per year. from 1999 to 2000: $\frac{55-39}{2000-1999} = 16$ percentage points per year.

(b) It makes the most sense to average the last two numbers, which gives us 14.5 percent per year.

(c) To estimate this way, you should first sketch a graph, and then draw a tangent line and figure out it slope. The answer should be roughly between 10 and 20, depending on your graph. The units are still percent per year.

□

§2.7 # 46. The number of bacteria in a controlled laboratory experiment after t hours is $n = f(t)$.

(a) What is the meaning of the derivative $f'(5)$?

(b) Suppose there is unlimited space and nutrients. Which do you think is larger, $f'(5)$ or $f'(10)$? If the nutrient supply is limited, does that change your answer?

Solution.

(a) The derivative is instantaneous rate of growth. So $f'(5)$ is how fast the bacteria population is growing after 5 hours.

(b) I think $f'(10)$ should be bigger. The rate at which the population grows should increase over time, since there will be more bacteria around to reproduce. If the nutrient supply is limited, however, then maybe after awhile bacteria will start dying, and the population will level off or decrease, so f' will be larger at smaller ts.

□

§2.7 # 49. The quantity of oxygen that can dissolve in water depends on the temperature of the water. The graph (in the book) shows how oxygen solubility S varies as a function of temperature T.

(a) What is the meaning of the derivative $S'(T)$? What are the units?

(b) Estimate the value of $S'(16)$ and interpret it.

Solution.

(a) the units of $S'(T)$ are mg per L per degree. So $S'(T)$ is telling you how much more oxygen can be dissolved in a liter of water as you instantaneously increase the temperature past T.

(b) From the graph, $S'(16)$ looks to be about $\frac{1}{4}$. That means that you can dissolve about $\frac{1}{4}$ mg more oxygen in a little of 16 degree water as make it hotter.