Math 1A — UCB, Fall 2010 — A. Ogus
Solutions[1] for Problem Set 1

§1.5 # 2.
(a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

(a) There is some real base a so that the tangent to a^x at the point $(0,1)$ has a slope of 1. e is defined to be that base.
(b) $e \approx 2.71828$
(c) e^x

$\square$

§1.5 # 17. Find the exponential function $f(x) = Ca^x$ whose graph is given.
Given that $f(x) = Ca^x$ and given the two points $(1,6)$ and $(3,24)$ on the graph of f, we obtain the following two equations:

$$24 = Ca^3 \tag{1}$$

$$6 = Ca^1 \tag{2}$$

Of course neither C nor a is equal to 0, so we may divide the first equation by the second, yielding $\frac{24}{6} = a^2$. Thus, $a = 2$. Substituting this value back into the second equation gives $C = \frac{6}{2} = 3$. Finally, we may conclude that $f(x) = 3 \cdot 2^x$.

$\square$

§1.5 # 20. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?

I One million dollars at the end of the month.
II One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the nth day.

Let's suppose the month has 30 days to begin with. Then the total money gained from option II is

$$1 + 2 + 4 + \cdots + 2^{29}\,\text{cents}$$

Just looking at the last of these numbers, a calculator gives

$$2^{29}\,\text{cents} = 536,870,912\,\text{cents} = 5,368,709.12\,\text{dollars}$$

Since 5,368,709.12 is larger than 1,000,000 (the total money gained from option I), we see that option II is preferable. However, what if the month in question only has 28 days? Still, since $2^{27}\,\text{cents} = 1,342,177.28\,\text{dollars} > 1,000,000\,\text{dollars}$, it is better to choose option II. Note that we haven't needed to calculate the total amount you would get through option II: it is sufficient just to establish a lower bound on the amount you would get through option II.

$\square$

§1.6 # 5.
Solution. It is 1-1 because every horizontal line intersects the graph at at most 1 point. $\square$

§1.6 # 6.
Solution. It is not 1-1 because the horizontal line $y = 0$ intersects the graph at 3 different points, i.e., the are 3 different values of x which satisfy $f(x) = 0$. $\square$

§1.6 # 17. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.
Solution. $g^{-1}(4)$ is the solution set of $3 + x + e^x = 4$. Obviously $x = 0$ is a solution, but $g(x)$ is strictly increasing, so $x = 0$ is the only solution. $g^{-1}(4) = \{0\}$. $\square$

§1.6 # 21. Find a formula for the inverse of the function $f(x) = \sqrt{10 - 3x}$.
Solution. Let $y = \sqrt{10 - 3x}$, square both sides, we get $y^2 = 10 - 3x$, so $x = \frac{10-y^2}{3}$. $\square$

§1.6 # 23. Find a formula for the inverse of the function $f(x) = e^{x^3}$.
Solution. Let $y = e^{x^3}$, take ln on both sides, we get $\ln y = x^3$, so $x = \sqrt[3]{\ln y}$. $\square$

§1.6 # 34. Find exact value of each expression.
(a) $\ln(1/e)$ (b) $\log_{10} \sqrt{10}$
Solution. (a) $\ln(1/e) = -\ln e = -1$.
(b) $\log_{10} \sqrt{10} = 1/2 \log_{10} 10 = 1/2$. $\square$

§1.6 # 35. (a) Find the exact value of $\log_2 6 - \log_2 15 + \log_2 20$.
Solution. $\log_2 6 - \log_2 15 + \log_2 20 = \log_2 2 + \log_2 3 - \log_2 3 - \log_2 5 + \log_2 4 + \log_2 5 = \log_2 2 + \log_2 4 = 1 + 2 = 3$. $\square$

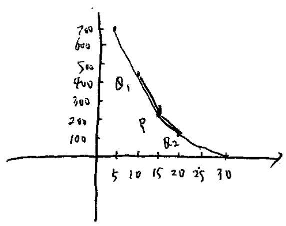
§1.6 # 54. Find (a) the domain of f and (b) f^{-1} and its domain where $f(x) = \ln(2 + \ln x)$.
Solution. (a) To make $\ln(2 + \ln x)$ meaningful, we should have $2 + \ln x > 0$, i.e., $\ln x > -2$, so the domain of f is $\{x : x > e^{-2}\} = (e^{-2}, \infty)$.
(b) The range of f is all the real numbers, so the domain of f^{-1} is the set of all real numbers. To get the formula for f^{-1}, let $y = \ln(2 + \ln x)$, so $e^y = 2 + \ln x$, we get $x = e^{e^y - 2}$. $\square$

§1.6 # 63. Find exact value of each expression.
(a) $\tan(\arctan 10)$ (b) $\sin^{-1}(\sin(7\pi/3))$
Solution. (a) $\tan(\arctan 10) = 10$ because $\tan \circ \arctan$ is the identity function from $\mathbb{R}$ to $\mathbb{R}$.
(b) $\sin^{-1}(\sin(7\pi/3)) = \sin^{-1}(\sin(2\pi + \pi/3)) = \sin^{-1}(\sin(\pi/3)) = \pi/3$ as $\pi/3$ lies between $-\pi/2$ and $\pi/2$. $\square$

§1.6 # 65. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.
Solution. Let $y = \sin^{-1} x$, then $\sin y = x$, we also have $\sin^2 y + \cos^2 y = 1$ and $\cos y \geq 0$ because y lies between $pi/2$ and $\pi/2$, so $\cos y = \sqrt{1 - x^2}$. $\square$

§2.1 # 1.
Solution. (a) The slope of the secant line PQ where $Q = (5, 694)$ is $\frac{694-250}{5-15} = -44.4$.
The slope of the secant line PQ where $Q = (10, 444)$ is $\frac{444-250}{10-15} = -38.8$.
The slope of the secant line PQ where $Q = (20, 111)$ is $\frac{111-250}{20-15} = -27.8$.
The slope of the secant line PQ where $Q = (25, 28)$ is $\frac{28-250}{25-15} = -22.2$.
The slope of the secant line PQ where $Q = (30, 0)$ is $\frac{0-250}{30-15} = -50/3$.
(b) The slope of the tangent line at $P \approx \frac{-38.8-27.8}{2} = -33.3$.
(c)

Solution to §2.1 1(c)



We use the average of slopes of PQ_1 and PQ_2 to estimate the slope of the tangent line at P.

we get -33.8 as in (b).

□

§2.1 # 3. The point $P(1, 1/2)$ lies on the curve $y = x/(1+x)$.

(a) If Q is the point $(x, x/(1+x))$, find the slope of the secant line PQ for the following values of x:

(i) 0.5 (ii) 0.9 (iii) 0.99

(b) Using the result of part (a), guess the slope of the tangent line to the curve at $P(1, 1/2)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(1, 1/2)$.

Solution. (a) (i) $\frac{1/2-0.5/(1+0.5)}{1-0.5} = 1/3 \approx 0.333333$.

(ii) $\frac{1/2-0.9/(1+0.9)}{1-0.9} \approx 0.263158$.

(iii) $\frac{1/2-0.99/(1+0.99)}{1-0.99} \approx 0.251256$.

(b) Guess the slope is 0.25 because as x gets closer and closer to 1, the slope of the secant line get closer and closer to 0.25.

(c) The equation of the tangent line is $y - 1/2 = 1/4(x-1)$. Simplify it, we get $y = 1/4x + 1/4$. □

§2.1 # 6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.

(a) Find the average velocity over the given time intervals:

(i) [1,2] (ii) [1,1.5] (iii) [1,1.1] (iv) [1,1.01] (v) [1,1.001]

(b) Estimate the instantaneous velocity when t=1.

Solution. (a) (i) The average velocity is $\frac{y(2)-y(1)}{2-1} = 4.42$

(ii) The average velocity is $\frac{y(1.5)-y(1)}{1.5-1} = 5.35$

(iii) The average velocity is $\frac{y(1.1)-y(1)}{1.1-1} = 6.094$

(iv) The average velocity is $\frac{y(1.01)-y(1)}{1.01-1} = 6.2614$

(v) The average velocity is $\frac{y(1.001)-y(1)}{1.001-1} = 6.27814$

(b) It is almost 6.27814. □