Math 1A — UCB, Fall 2009 — M. Christ
Solutions[1] to problem Set 14

§6.1 # 4. Find the area of the shaded region.
Solution: The area is $\int_0^3 [2y - y^2 - (y^2 - 4y)]dy = \int_0^3 2y - 2y^2 dy = 2(\frac{1}{2}3^2 - \frac{1}{3}3^3) = 3$

§6.1 # 13. Find the area of the region between the functions $y = 12 - x^2$ and $y = x^2 - 6$.
Solution: First we find the two points of intersection, in order to know what interval to integrate over: $12 - x^2 = x^2 - 6$ if and only if $x^2 = 9$, so $x = 3, -3$. We integrate with respect to x. Thus the area is $\int_{-3}^3 (12 - x^2 - (x^2 - 6))dx = \int_{-3}^3 18 - 2x^2 dx = 108 - 2(\frac{1}{3}3^3 - \frac{1}{3}(-3)^3) = 72$

§6.1 # 19. Find the area of the region between the functions $x = 2y^2$ and $x = 4 + y^2$.
Solution: We will integrate with respect to y. Points of intersection: $2y^2 = 4 + y^2$, $y = 2, -2$. The area is $\int_{-2}^2 (4 + y^2 - 2y^2)dy = \int_{-2}^2 (4 - y^2)dy = 16 - \frac{16}{3} = \frac{32}{3}$

§6.1 # 31. Evaluate the integral and interpret it as the area of a region $\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x| dx$.
Solution: It is the area betweent the graphs of the function $y = \sin x$ and $y = \cos(2x)$. To eliminate the absolute value signs, we need to find out at which areas $\sin x - \cos 2x$ is positive and at which negative. Since the function is continuous, let us first find the zeros, i.e. when $sinx = cos(2x)$. $cos(2x) = 1 - 2sin^2 x$. So $sinx = \frac{1}{2}$ is a solution, i.e. if $x = \frac{\pi}{6}$. Looking at the sketch, we see that the function is positive if $0 < x < \frac{\pi}{6}$ and negative if $\frac{\pi}{6} < x < \frac{\pi}{2}$. Thus $\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x| dx = \int_0^{\frac{\pi}{6}} \sin x - \cos 2x dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2x - \sin x dx = \frac{3}{2}\sqrt{3} - 1$.

§6.1 # 40. Sketch the region in the xy-plane defined by the inequalities $x - 2y^2 \geq 0$ and $1 - x - |y| \geq 0$ and find its area
Solution: The first inequality gives us $x \geq 2y^2$, so if we draw the curve $x = 2y^2$, every point in this region has to lie on this curve or to the left of this curve. Similarly, if we consider the two cases (y positive and y negative) of $1 - x \geq |y|$, we find that the region is bounded by the three functions $x = 2y^2$, $x = y + 1$ and $x = 1 - y$. The intersection points have y values $\frac{1}{2}, -\frac{1}{2}$. By symmetry, the area is $2\int_0^{\frac{1}{2}} (1 - y - 2y^2)dy = 2(\frac{1}{2} - \frac{1}{8}\frac{1}{12}) = \frac{7}{12}$

§6.1 # 43. Use the midpoint rule to estimate the area.
Solution: 40(20.3+29.0+27.3+20.5+8.7) = 4232.

§6.1 # 45. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity.
Solution: (a) After one minute, A is ahead, since it has always been driving with higher speed.
(b) The area represents the distance between the two cars after 1 minute.
(c) Car A is still ahead, because the shaded area is greater than the area between the curves integrated over time 1 to 2.
(d) They are side by side, when the integral $\int_0^t f_A - f_B dt = 0$, which happens at about $t = 2.2$.

§6.2 # 7. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$ and $y = x$, for $x \geq 0$ around the x-axis.

Solution. Drawing a picture helps. The curves intersect when $x^3 = x$, so when $x = -1, 0, 1$. We care about the region from $x = 0$ to $x = 1$, as a result. Remember that in this region, $x > x^3$. We have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - (x^3)^2)\,dx = \pi\left(\frac{1}{3}x^3 - \frac{1}{7}x^7\right)\Bigg|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{4\pi}{21}$$

□

§6.2 # 12. Find the volume of the solid obtained by rotating the region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$ about the line $y = 2$.

Solution. Again, draw a picture. Then

$$\begin{aligned}
V = \int_0^2 A(x)\,dx &= \int_0^2 \pi[(1 + (1 - e^{-x}))^2 - 1^2]\,dx \\
&= \int_0^2 \pi(3 - 4e^{-x} + e^{-2x})\,dx \\
&= \pi\left(3x + 4e^{-x} - \frac{1}{2}e^{-2x}\right)\Bigg|_0^2 \\
&= \pi(6 + 4e^{-2} - \frac{1}{2}e^{-4} - 4 + \frac{1}{2}) \\
&= \pi\left(\frac{5}{2} + 4e^{-2} - \frac{1}{2}e^{-4}\right)
\end{aligned}$$

□

§6.2 # 28. Refer to the figure and find the volume generated by rotating the given region about the specified line. The line specified is $\mathcal{R}_3$ about the line OA, which is the x-axis.

Solution. Inspection of the picture shows that

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(\sqrt{x}^2 - (x^3)^2)\,dx = \pi\left(\frac{1}{2}x^2 - \frac{1}{7}x^7\right)\Bigg|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{7}\right) = \frac{5\pi}{14}$$

□

§6.2 # 49. Find the volume of a right circular cone with height h and base radius r.

Solution. The easiest way to do this is to rotate a triangle of width h and base height r around the x axis. Such a triangle is formed by $x = 0$, $y = 0$, and the line $y = r - rx/h$. Then

$$V = \int_0^h \pi(r - rx/h)^2\,dx = \frac{-\pi h}{3r}(r - rx/h)^3\Bigg|_0^h = 0 - \frac{-\pi h}{3r}r^3 = \frac{1}{3}\pi h r^2$$

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□

§6.2 # 51. The the volume of the cap of a sphere with radius r and height h.

Solution. The region can be thought of as the product of rotating the portion of the circle $x^2+y^2 = r^2$ bounded by $x = 0$ and $y = 0$ around the y axis. The best way to do this is to write the circle as a function of y, so $x = \sqrt{r^2 - y^2}$. Then

$$\begin{aligned}
V &= \int_{r-h}^{r} A(y)\, dy \\
&= \int_{r-h}^{r} \pi(\sqrt{r^2 - y^2})^2\, dy \\
&= \pi\left(r^2 y - \frac{1}{3}y^3\right)\Big|_{r-h}^{r} \\
&= \pi\left(r^3 - \frac{1}{3}r^3 - r^2(r-h) + \frac{1}{3}(r-h)^3\right) \\
&= \pi\left(h^2 r + \frac{1}{3}h^3\right)
\end{aligned}$$

□

§6.2 # 65(b). Use Cavalieri's Principle to find the volume of an oblique cylinder with radius r and height h.

Solution. The idea is that this solid has cross-sectional areas (all of which are πr^2) equal to those of a regular cylinder of radius r and height h, whose volume is known to be $V = \pi r^2 h$. So the volume of the specified oblique cylinder is the same. This volume can also be calculated by rotating a rectangle about the y- or x-axis.

□