Solutions for Midterm Exam 1

There were three versions of this exam, with slightly different numbers and/or functions. For most problems I’ll give a solution for only one of the three versions.

(1a) Use limit rules to evaluate \( \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \).

**Solution.** For any \( x > 0 \) which is not equal to 9,

\[
\frac{\sqrt{x} - 3}{x - 9} = \frac{\sqrt{x} - 3}{\sqrt{x} + 3} = \frac{x - 3}{(x - 3)(\sqrt{x} + 3)} = \frac{1}{\sqrt{x} + 3}.
\]

Thus

\[
\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \lim_{x \to 9} \frac{1}{3 + 3} = \frac{1}{6},
\]

using the sum, ratio, and direct substitution rules for limits. Note that when I used the ratio rule, the limit of the denominator was not 0.

(1b) Let \( f(x) = \sqrt{x} \), with its natural domain. Does \( f'(9) \) exist?

**Solution.** Yes. The definition of the derivative says that \( f'(9) \) exists if \( \lim_{x \to 9} \frac{f(x) - f(9)}{x - 9} \) exists (and is a real number). In problem (1a) we showed that this limit does exist.

(2a) Let \( f(x) = \frac{(x-2)(x-4)(x-8)}{3(x-1)(x-2)(x-8)} \), with its natural domain. Find all asymptotes of the graph of \( f \).

**Solution.** Vertical asymptote at \( x = 1 \). Two horizontal asymptotes at \( y = \frac{1}{3} \) as \( x \to \infty \) and as \( x \to -\infty \). (There are no asymptotes at \( x = 2 \) and \( x = 8 \).)

(2b) Find \( \lim_{x \to 0} x^2 \sin(e^{1/x}) \).

**Solution.** The limit is zero. Let \( g(x) = x^2 \sin(e^{1/x}) \), \( f(x) = -x^2 \), \( h(x) = x^2 \). Then since \( \sin(y) \in [-1,1] \) for any real number \( y \), \( f(x) \leq g(x) \leq h(x) \) for all \( x \neq 0 \). We know that \( \lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} h(x) \), by the direct substitution property. Therefore the Squeeze Theorem says that \( \lim_{x \to 0} g(x) \) also exists and equals 0.

(3) Show that there is at least one real number \( x \) which satisfies \( x^6 = 1 + \sin(x) \).

**Solution.** Define \( f(x) = x^6 - 1 - \sin(x) \) for all real numbers \( x \). We need to show that \( f(x) = 0 \) for at least one \( x \). \( f \) is a continuous function, since sums of continuous functions are continuous, and all the basic functions (including polynomials and \( \sin \)) are continuous. Consider the interval \( [a,b] = [0,2] \). \( f(a) = 0 - 1 - 0 < 0 \), while \( f(b) = 2^6 - 1 - \sin(2) \geq 2^6 - 1 - 1 = 62 > 0 \).

Therefore the Intermediate Value Theorem applies and says that \( f(x) = 0 \) for at least one \( x \) in \( [0,2] \).

(4a) Let \( r > 0 \). How is \( \log_r(2) \) defined?

**Solution.** It is the unique real number \( x \) which satisfies \( r^x = 2 \). (Full credit for that answer.)

(A more complete answer might also include: \( \log_r \) is the inverse of the function \( r^x \). The range of \( r^x \) is \( (0,\infty) \), so the domain of \( \log_r \) is \( (0,\infty) \); thus 2 belongs to its domain.)
(4b) Let \( f(x) = \tan(x) \) with domain \((-\frac{\pi}{2}, 0)\). Does \( f \) have an inverse? If so, what are the domain and range of the inverse function?

**Solution.** For \( x \in (-\frac{\pi}{2}, 0) \), \( \cos(x) \neq 0 \) so \( \tan(x) = \frac{\sin(x)}{\cos(x)} \) is defined. \( \tan \) is an increasing function which takes on all values in \((-\infty, 0)\) as \( x \) varies over \((-\frac{\pi}{2}, 0)\). Thus \( f \) has an inverse function. Its domain is \((-\infty, 0)\), and its range is \((-\frac{\pi}{2}, 0)\).

(4c) If some *vertical* line intersects a graph at more than one point, what does this say about the graph?

**Solution.** The graph is not the graph of any function.

(4d) Simplify: \( \ln(5e\sqrt{x}) \), assuming that \( x > 0 \).

**Solution.** \( \ln(5e\sqrt{x}) = \ln(5) + \ln(e) + \ln(\sqrt{x}) = \ln(5) + 1 + \frac{1}{2} \ln(x) \).

(4e) If the domain of \( f \) contains \((-1, 1)\), and if \( f \) is continuous at 0, must \( f'(0) \) exist?

**Solution.** No. Example: \( f(x) = |x| \). \( f(x) = \sqrt{|x|} \) is also an example. Full credit for any correct example, of course, whether we learned it in this course or not.

(5a) Let \( f(x) = x^2 \). Find \( \delta > 0 \) such that \( |f(x) - 36| < \frac{1}{1000} \) whenever \( |x - 6| < \delta \).

**Solution.** It will be useful to know that \( |f(x) - 36| = |x^2 - 36| = |x - 6| \cdot |x + 6| \).

Define \( \delta = \frac{1}{13,000} \). Note that \( \delta < 1 \). Therefore if \( |x - 6| < \delta \) then \( x \in (5, 7) \) and thus \( |x + 6| < 13 \).

Therefore if \( |x - 6| < \delta \) then
\[
|f(x) - 36| \leq |x - 6| \cdot |x + 6| \leq \delta \cdot 13 < \frac{13}{13,000} = \frac{1}{1000}.
\]

(5b) Show, using the precise definition of a limit, that
\[
\lim_{x \to \frac{1}{3}} (9x - \frac{1}{x}) = 0.
\]

**Solution.** It will be useful to know that for any \( x \neq 0 \),
\[
|9x - \frac{1}{x}| = \left| \frac{9x^2 - 1}{x} \right| = \frac{|3x - 1| \cdot |3x + 1|}{|x|} = \frac{3|x - \frac{1}{3}| \cdot |3x + 1|}{|x|}.
\]

Let any \( \varepsilon > 0 \) be given. Define \( \delta = \min\left(\frac{1}{6}, \frac{\varepsilon}{54}\right) \). If \( |x - \frac{1}{3}| < \delta \) then \( \frac{1}{6} < x < \frac{1}{2} \), so \( 1/|x| \leq 6 \), and \( |3x + 1| \leq \frac{3}{2} + 1 = \frac{5}{2} < 3 \).

Therefore if \( 0 < |x - \frac{1}{3}| < \delta \) then
\[
|9x - \frac{1}{x}| < 3 \cdot 6 \cdot 3 \cdot |x - \frac{1}{3}| < 54\delta.
\]

Since \( \delta \leq \varepsilon/54 \), this is \( \leq \varepsilon \), and therefore \( |9x - \frac{1}{x}| < \varepsilon \).

(Full credit would be given for \( \delta = \min\left(\frac{1}{6}, \frac{\varepsilon}{36}\right) \).)