Solutions for some homework problems

7.4, 3: Find a 2-Sylow subgroup and a 3-Sylow subgroup of S_4.

Solution: S_4 has 24 elements, so a 2-Sylow subgroup will have order 8 and a 3-Sylow subgroup will have order 3. The subgroup H of S_4 generated by (1 2 3 4) and (1 3) has order 8, and is thus a 2-Sylow subgroup. (Note that (1 3) belongs the normalizer of the subgroup generated by (1 2 3 4), which shows that H has order 8. Note also that H is isomorphic to D_4.) The subgroup of S_4 generated by (1 2 3) has order 3 and is thus a 3-Sylow subgroup.

7.4, 9: Let G be a group of order 148. Show that G is not simple.

Solution: $148 = 4 \times 37$. By Sylow's theorem, it has at least one subgroup P of order 37. If P' is another, then $P \cap P'$ is just the identity, since its order must properly divided the prime number 37. Then the map $P \times P' \to G$ is injective, which is not possible, since 37^2 is larger than 148. In particular, every conjugate of P is again just P, so P is normal and G is not simple.

7.4, 11: Let G be a group of order p^2q, where p and q are distinct primes. Show that G is not simple.

Solution: First suppose that $q < p$. Then a p-Sylow subgroup of G has index the smallest prime dividing $|G|$, and hence is normal by problem 12 of section 7.3. So suppose that $q > p$. Recall that the number n_q of q-Sylow subgroups Q is congruent to 1 modulo q and divides the index of Q in $|G|$, which in this case is p^2. So the only possibilities are $1, p$, and p^2. If $n_q = 1$, Q is normal, and we are done. Since $p < q$, we can't have $p \equiv 1 \pmod{q}$. If $n_q = p^2$, there are p^2 subgroups of order q, and their only intersection is in the identity. This gives us $p^2(q-1)$ elements of exact order q, leaving only p^2 elements remaining in the group. But any p-Sylow subgroup P must then consist of all these remaining elements. This implies that P is unique, hence normal, so the again G is not simple.